THE AMERICAN MATHEMATICAL MONTHLY

THE MATHEMATICAL ASSOCIATION OF AMERICA

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THE SIXTH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The sixth regular meeting of the Southern California Section of the Mathematical Association of America was held at Occidental College, Los Angeles, on Saturday, Nov. 5, 1927. Professor W. P. Russell presided.

There were fifty present, including the following thirty seven members of the Association: O. W. Albert, E. E. Allen, L. D. Ames, M. A. Basoco, H. Bateman, Clifford Bell, E. T. Bell, W. N. Birchby, J. R. Campbell, M. Collier, P. H. Daus, H. H. Gaver, H. E. Glazier, E. R. Hedrick, H. C. Hicks, G. H. Hunt, G. James, M. M. Keith, Deca Lodwick, W. E. Mason, Ada McClellan, G. F. McEwen, B. Podolsky, A. Prater, M. Reinert, L. E. Reynolds, W. P. Russell, G. E. F. Sherwood, H. M. Showman, D. V. Steed, P. W. Stoner, H. C. Van Buskirk, M. Ward, L. E. Wear, H. C. Willett, Clyde Wolfe, E. R. Worthington.

The meeting began with a luncheon at the college commons, and the Association was welcomed to Occidental College by Acting President, Robert Cleland. After the luncheon, the meeting adjourned to room 28, Fowler Hall for the following program. The numbers of the abstracts correspond to the numbers of the papers.

1. "The relation between certain functions analogous to hyperbolic functions and the Pellian cubic." by Professor P. H. DAUS, University of California at Los Angeles.

2. "The solution of linear differential equations in series by operational divisors," by Mr. MORGAN WARD, California Institute of Technology.

3. "Descriptive geometry of four dimensional space," by Professor L. D. AMES, University of Southern California.

4. "Some aspects of graduate instruction in mathematics" by Professor E. T. BELL, California Institute of Technology.

5. "Suggestions regarding the drawing of plane curves" by Professor E. R. HEDRICK, University of California at Los Angeles.

1. By making certain changes in an article by L. E. Ward in the June-July 1927 issue of this Monthly, Professor Daus considered certain functions analogous to hyperbolic functions, and discussed an isomorphism between them and the solutions of the Pellian cubic, $X^3+DY^3+D^2Z^3-3DXYZ=1$.

2. The connection between the ordinary method of solving a linear differential equation in series and the operational method was given and the practicability of the latter discussed.

3. Descriptive geometry furnishes a method by which space of three dimensions can be represented on a plane. The methods used are easily generalized. An obvious generalization of familiar principles of perspective can be used to represent n-dimensional aggregates on an aggregate of $n-1$ dimensions.

Professor Ames showed a three dimensional perspective model of a certain four dimensional aggregate. The application of these principles seems to involve no essential difficulties not met in three dimensions.

4. The speaker reported on changing conditions in graduate instruction in mathematics, owing to greatly increased enrollment, and outlined probable modifications in the curriculum leading to the doctor's degree.

5. Professor Hedrick discussed certain methods of tracing curves, which involve at most, the use of the fundamental drawing instruments. Besides the well known methods of addition, multiplication and division discussed in texts on Analytic Geometry, he discussed the mechanical construction of a function of a function exhibiting how such curves can be quickly traced. This paper was followed by a discussion by Professors Albert, Stoner, and Wolfe.

P. H. DAUS, *Secretary.*

MATHEMATICS FOR STUDENTS OF CHEMISTRY[1]

By FARRINGTON DANIELS, Department of Chemistry, University of Wisconsin.

1. Introduction. Chemistry has graduated from the class of descriptive sciences into the class of exact sciences and has taken its place by the side of physics and engineering as a branch of mathematics. All chemical phenomena and many biological phenomena are now being interpreted in terms of physical chemistry, and physical chemistry cannot be mastered without a foundation in mathematics which extends through calculus.

Research chemists are keen to realize their dependence on mathematics and many of them feel blocked in their progress by their weakness in mathematics. Another illustration of the need for mathematics is the fact that an increasing number of graduate students in chemistry are taking mathematics as a minor.

Unfortunately the undergraduate student and his advisers do not realize the necessity for mathematics until it is too late. The one overwhelming problem in the teaching of physical chemistry is the handicap of inadequate prepa-

[1] Presented at the meeting of the American Mathematical Association in Madison, Wis., Sept. 6, 1927. The material was presented with the help of special lantern slides. In making these slides, the writing was done with India ink and a smooth pen on an old kodak film which had been soaked and scraped to remove the gelatine film. The film was then mounted between two cover glasses, like those ordinarily used for lantern slides, and bound with gummed paper. The expense and time of a photographic process was thus eliminated. The cover glasses and the binding paper for ordinary lantern slides may be purchased for a few cents at any photographic store. In case many slides are to be made, thin transparent sheets of "cellophane" may be used in place of the kodak films. This material is now used in the confectionery trade and may be purchased very cheaply. It should be cut to the proper size, stretched over the edges of one of the glass plates while wet, and allowed to dry.

ration in mathematics. Calculus should be a prerequisite for every course in physical chemistry but in many cases it is not a prerequisite. Only the simplest differentiations and integrations are used but the ability to think in mathematical terms is absolutely essential.

A few specific applications of mathematics in physical chemistry are given in the following sections. They have been chosen either because they are particularly important or because experience has shown that they constitute stumbling blocks for the student.

2. Calculus. Simple differentiations and simple integrations are common in physical chemistry, and integration tables are freely used. The nature of physical chemical experiments with their complicated apparatus is such as to demand a close personal contact between student and instructor and when a student begins to fall behind the first question always is: "Have you had calculus?" There seem to be but three replies: "No"; "Yes, but it was a long time ago."; and "I was exposed but it didn't take." Now in most cases the preparation is ample for the demands of physical chemistry but the students seem to lack confidence. Can it be that they think that they are expected to remember all the half hundred rules of calculus? The principles of calculus rather than the details of calculus are needed in understanding physical chemistry and only the simplest parts of calculus are actually used.

3. The compound interest law:

$$k = \frac{2.303}{x_2 - x_1} \log_{10} \frac{y_1}{y_2}.$$

This equation is used extensively in the treatment of reaction rates, the absorption of light, the cooling of heated bodies and in many other phenomena. The logarithmic form is usually used rather than the exponential form $y = y_0 e^{-kx}$.

4. The exponential equation:

$$y = e^{\frac{-2500}{2T}}.$$

There is no surer way to make the average industrial chemist skip a page of a scientific article than to place on it a few exponential equations of this type. Practice in transforming and solving these equations is very necessary.

5. Partial differentiation. Chemical thermodynamics is largely founded on partial differentiation and the following factors may be involved: pressure, temperature, volume, entropy, heat, work, internal energy, heat content, free energy, thermodynamic potential. In thermodynamic systems involving any three of the ten variables there are 720 partial first derivatives and there are

11 billion thermodynamic relations involving any four derivatives. Bridgman[1] has shown how these may be reduced to 45 types and only a half dozen of these are important in chemistry.

6. Statistical mechanics is just entering the field of chemistry, and research chemists feel that with this tool will be found the solution of many of the unsolved problems of physical chemistry. In practically no case does the chemist's precision extend beyond a millionth of a gram molecule of material or 6×10^{17} molecules. There is always, then, a sufficiently large number of individuals to assure the application of the laws of probability with accuracy. An important book by Tolman,[2] just published, emphasizes clearly the value of statistical mechanics in chemical problems.

7. Errors in experimental measurements have been treated with the help of probability theorems and the average error, the probable error, and the root-mean-square error have been used. Only a few of the measurements of physical chemistry are sufficiently accurate to warrant the use of more than the average error, because temperature fluctuations and constant errors are usually more serious than the errors of measurement.

8. The differential equations:

$$(a)\qquad \frac{(d\log_e k)}{dT}=\frac{Q}{(2T^2)}\,; \qquad (b)\qquad dx/dt = k(a-x)(b-x).$$

Many of the differential equations of physical chemistry are so simple that the student does not realize that he is solving differential equations.

Equation (a) in which Q is assumed to be independent of T gives the influence of temperature on a great many chemical phenomena such as chemical equilibrium, rate of reaction, solubility, vapor pressure, and others.

Equation (b) involves the rate of reaction between two different substances, and there are cases of several reactions occurring simultaneously which are more difficult. Differential equations are sufficiently important in chemistry to warrant the publication of a special book.[3]

9. Algebra. The algebraic transformations illustrated by the following examples are much used.

(a) *If* $P_1 - P_2 = P_2C_v/(C_p - C_v)$ then $P_1/(P_1 - P_2) = C_p/C_v$.

$$(b)\qquad \left(\frac{2x}{1+x}\right)^2 \div \frac{1-x}{1+x} = \frac{4x^2}{1-x^2}.$$

[1] Bridgman, *A Condensed Summary of Thermodynamic Formulae*, Harvard University Press, Cambridge, Mass., 1925.

[2] Tolman, *Statistical Mechanics with Applications to Physics and Chemistry*, Chemical Catalog Co., New York, 1927.

[3] Hitchcock and Robinson, *Differential Equations in Applied Chemistry*, Wiley and Sons, New York, 1924.

If the algebraic process is pointed out, the student understands these steps perfectly; but few students take the initiative in proving these steps as they occur in the derivation of chemical formulae.

10. Quadratic equations are avoided by students whenever possible, except by engineers and those who have worked many problems. The following equation is typical:

$$K = \tfrac{1}{4}x^{-2}(a - x)(b - x)\ ; \quad x = ?$$

11. Large and small numbers. Molecules in a gram molecule $= 6.06 \times 10^{23}$; the charge on the electron $= 1.59 \times 10^{-19}$; Planck's constant $= 6.56 \times 10^{-27}$; the wave length of blue light $= 4 \times 10^{-5}$; the frequency of blue light $= 7.5 \times 10^{14}$.

These numbers were not necessary in the descriptive chemistry of the past but they are absolutely necessary now. Important advances have been made by visualizing the individual units of matter, electricity and radiation, the molecule, the electron and the quantum.

Twenty years ago it was taught that oxygen is sixteen times as heavy as hydrogen but that the absolute weights could not be known. Now it is known with precision that one atom of oxygen weighs $16/(6.06 \times 10^{23})$ gram. Again, visualizing the process of electrolysis, an ion receives an electron and is discharged as an atom, and the number of atoms (6.06×10^{23}) multiplied by the charge on one electron (1.59×10^{-19}) gives 96,500 coulombs, a quantity which was formerly considered an empirical constant in Faraday's law.

Students are afraid of these large and small numbers and it would be desirable if they could acquire familiarity with them in elementary courses by solving problems such as "How many millimeters to the moon?" or "How many molecules of water in Lake Michigan?" Students seem to enjoy problems of this kind.

12. Negative logarithms.

(*a*) $$10^{-5.7} = x, \quad x = ?$$

(*b*) $$.0592 \log 0.0093 = \frac{0.3984}{x}, \quad x = ?$$

Problems of this kind are very common in electro-chemistry and particularly in the treatment of acid and alkaline solutions. If x in equation (*a*) represents the concentration of hydrogen ions, 5.7 is the so-called pH of the solution. This term is universally used in the biological and botanical sciences, but not one student in five can solve these problems.

13. The slide rule. The slide rule is sufficiently accurate for most chemical calculations and it develops judgment in matters of accuracy. For example, when students titrate 30.0 cc of alkali against 40.0 cc of standard acid, the one

who uses a slide rule and is accustomed to evaluating the decimal point writes down a normality factor of 1.333. A routine calculator, using long division or logarithms, may write down 1.333333 and imply an accuracy far greater than is justified.

14. Displaced origins and unequal scales. Most students can use graph paper satisfactorily but there are always a few who insist on starting the scales at zero and wasting most of the paper. If the volume of gas is to be plotted against temperatures between 273 and 293, there are some who will start with zero rather than with 270 or 273. Many students have difficulty in finding the equation of a line unless the divisions along the x-axis are equal to those of the y-axis. It would be better if the elementary books didn't always start the scales at zero and didn't always have the same scales along the two axes.

15. Finding empirical equations.

$$(a)\quad y = a + bx + cx^2\,; \qquad (b)\quad \log y = a + bx.$$

Equations of these two types are used very often to express experimental data. The first one is purely empirical but the second one has a theoretical backing and it runs through a great many phenomena of physical chemistry. If students of analytical geometry could have more direct experience in curve fitting and in evaluating the constants of these equations from a set of data, it would help greatly in the sciences.

16. Methods for expressing data. The experimental determination of the vapor pressure of chloroform at different temperatures illustrates nicely the advantage of mathematical treatment. The student obtains the following data in the laboratory:

Temperature	(T)	293°	303	313	323	333
Vapor Pressure	(P)	160.5mm	247.5	369.3	535.0	755.5

He then plots vapor pressure against temperature and obtains an exponential curve. He next plots $\log P$ against $1/T$ and obtains a line which is nearly straight. Finally, he finds the equation of this straight line and writes

$$\log P = \frac{-1641.2}{T} + 7.8089.$$

The student gets real pleasure in work of this kind and he is convinced that the mathematical equation is much the best method for expressing the relation between variables.

17. Graphical calculations. A science which has recently become a mathematical science must limp along with graphical calculations for a while. For example, there are few graphical calculations left in astronomy, while in economics, a subject which involves more factors than chemistry, the graphical methods are still more prevalent than in chemistry.

There is at present a great need for more precise graphical methods, or for reasonably simple substitutes for the graphical calculations.

In many cases there are too many factors involved in the phenomenon to offer any immediate hope of finding a theoretical equation connecting the variables, but from a single graph many other properties may be calculated with an accuracy which depends only on the precision of the graphical method.

G. N. Lewis has emphasized the value of graphical calculations in chemistry and devised methods for increasing their accuracy. The methods are included in his book on thermodynamics.[1] The calculations consist chiefly in finding the tangents to a curve or in evaluating the area under a curve.

The velocity of a chemical reaction is determined by plotting the amount of material against the time and determining the tangent to the curve. The properties of a dissolved substance in solution are now defined in terms of the rate of change of the property with the change in concentration. For example, the partial volume of a salt in water is determined by plotting the volume of the solution against the concentration and drawing a tangent at any specified concentration. This obviously simple method for expressing the properties of a solution was devised by Lewis only a few years ago and it brought order out of chaos in the study of solutions. Previously, different investigators had worked over different finite concentrations and their results could not check. Ingenious methods have been devised for increasing the accuracy of these calculations. In one method for example the difference between the true differential value and an approximate finite value is plotted, and any error of graphing applies only to a small correction term rather than to the quantity itself.

When the specific heat of a substance is plotted against the temperature, the area under the curve down to absolute zero is a direct measure of the "entropy" of the substance, and the entropy change in a chemical reaction is a measure of the tendency to react. Chemical processes will soon be calculated from tables of entropy just as engineering problems are now solved with the help of tables. In accumulating the material for these entropy tables from experimental measurements the graphical methods must play an important part because the equations for the specific heat curves are unknown except in the region of the absolute zero.

Many of the calculations of physical chemistry may be made exact instead of approximate by substituting "activities" for concentration, and the activity of a dissolved substance may be calculated from the activity of the dissolving liquid by evaluating the area under a curve which is plotted from experimental data. Frequently these curves approach the axes asymptotically in such a

[1] Lewis and Randall, *Thermodynamics and the Free Energy of Chemical Substances*, McGraw-Hill Book Co., New York, 1923.

way as to render the calculation difficult, and special functions have been invented for the graphing, to avoid these inconvenient figures and permit an accurate determination of the area.

Any advances which permit an increased accuracy in graphical calculations will be welcomed by the chemist.

18. Conclusion. It has been the purpose of this article to point out some of the mathematical needs of modern chemistry. It would be appreciated if courses in mathematics could emphasize some of these things, along with the calculations of volumes, lengths of curves, and moments of inertia and other calculations which are included for the benefit of the engineer. The present courses should not be changed, however, for mathematics is much more valuable to the student of chemistry as a mental training than as a source of technical methods. In physical chemistry the chief aim is to emphasize the research point of view and to interest the student in the mechanism of natural phenomena and there is no better way to develop the necessary originality in a student than to have him solve hundreds and thousands of problems in pure mathematics.

A SIMPLIFICATION OF CERTAIN PROBLEMS IN ARITHMETICAL DIVISION

By MORGAN WARD, California Institute of Technology

1. Like all inverse operations, the process of dividing one integer by another requires certain tentative steps irresolvable into the direct operations of addition and multiplication. Any method of division has then a three-fold aim: to reduce as far as possible (1) the number of these tentative steps, (2) the difficulty of each step, and (3) the amount of addition, multiplication and mere copying necessary to combine the results of the separate steps into the correct quotient. These aims conflict to a certain extent and our ordinary "long" division is a sort of compromise between them. It may in fact be considerably improved as regards the third requirement.[1]

The object of this paper is to exhibit a method of division satisfying the requirements above and applicable to a class of problems where solution by ordinary division is excessively laborious, if not impossible. Our main result is given in section 4; its proof, which rests upon the most elementary properties of congruences and power residues, is developed in sections 2 and 3. The concluding section is devoted to numerical examples.

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[1] See a discussion of "long" division by L. S. Dederick in this Monthly, vol. 33 (1926), pp. 143–144.

2. In what follows small italic and Greek letters stand for positive integers or zero.

Denote the quotient obtained on dividing a by $b \neq 0$ by

$$(1) \qquad [a/b] = q \quad \text{so that} \quad a = qb + r \qquad 0 \leqq r < b.$$

Then $q = 0$ when $b > a$ and is otherwise a positive integer. It is to be understood that b is never zero in the symbol $[a/b]$. Let us develop some of the properties of the symbol.

First, if $b = b_1 \cdot b_2$, it is easily shown[1] that

$$(I) \qquad \left[\frac{a}{b_1 b_2}\right] = \left[\frac{\left[\frac{a}{b_1}\right]}{b_2}\right] = \left[\frac{\left[\frac{a}{b_2}\right]}{b_1}\right]$$

Suppose $a = a_1 \cdot a_2$. Let $[a_1/b] = q_1$ and $[a_2/b] = q_2$ so that $a_1 = bq_1 + r_1$ and $a_2 = bq_2 + r_2$ $(0 \leqq r_1, r_2 < b)$. Then $a_1 a_2 = (bq_1 q_2 + q_1 r_2 + q_2 r_1)b + r_1 r_2$. Hence

$$(II) \qquad \left[\frac{a_1 a_2}{b}\right] = b\left[\frac{a_1}{b}\right]\left[\frac{a_2}{b}\right] + r_2\left[\frac{a_1}{b}\right] + r_1\left[\frac{a_2}{b}\right] + \left[\frac{r_1 r_2}{b}\right].$$

The following special cases of II are to be noted:

First, if $r_2 = 0$,

$$(IIa) \qquad \left[\frac{a_1 a_2}{b}\right] = b\left[\frac{a_1}{b}\right]\left[\frac{a_2}{b}\right] + r_1\left[\frac{a_2}{b}\right] = a_2\left[\frac{a_1}{b}\right] + r_1\left[\frac{a_2}{b}\right].$$

Secondly, if $q_2 = 0$, then $a_2 < b$, $[a_2/b] = 0$, $r_2 = a_2$, and

$$(IIb) \qquad \left[\frac{a_1 a_2}{b}\right] = a_2\left[\frac{a_1}{b}\right] + \left[\frac{a_2 r_1}{b}\right].$$

Also if $a = a_1 + a_2$,

$$(III) \qquad \left[\frac{a}{b}\right] = \left[\frac{a_1}{b}\right] + \left[\frac{a_2}{b}\right] + \left[\frac{(r_1 + r_2)}{b}\right], \quad \text{where}$$

$$0 \leqq \left[\frac{(r_1 + r_2)}{b}\right] \leqq 1.$$

3. Assume now that a is greater than b and prime to it,

$$(2) \qquad U_m = \left[\frac{a^m}{b}\right]. \quad \text{Then} \quad (2a) \quad U_0 = U_1 = 0.$$

We seek the general expression for U_m.

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[1] Reid: *Elements of the Theory of Algebraic Numbers*, p. 27.

Let r be the least positive integer for which

$$a^{r+1} \equiv a \pmod{b}. \tag{3}$$

That is, r is the exponent to which a belongs, modulo b, so that $a, a^2, a^3, \cdots, a^r$ are all distinct. Let $a^s \equiv a_s \pmod{b}$ $(0 \leqq a_s < b)$ and

$$V_s = [aa_s/b], \quad \text{so that} \quad V_0 = 0, \quad V_s < b. \tag{4}$$

Then by (3), (4), $V_s = V_t$ when and only when $s \equiv t \pmod{r}$.

By (IIb), $U_m = [a \cdot a^{m-1}/b] = a[a^{m-1}/b] + [a \cdot a_{m-1}/b]$, $m \geqq 1$; or, by (2) and (4),

$$U_m = aU_{m-1} + V_{m-1}. \tag{5}$$

Thus $U_2 = aU_1 + V_1 = V_1$; $U_3 = aU_2 + V_2 = aV_1 + V_2$. Assume for some $s \geqq 2$

$$U_s = a^{s-2}V_1 + a^{s-3}V_2 + \cdots + aV_{s-2} + V_{s-1} = \sum_{K=1}^{s-1} a^{s-1-K}V_K.$$

Then

$$U_{s+1} = aU_s + V_s = \sum_{K=1}^{s-1} a^{s-K}V_K + V_s = \sum_{K=1}^{s} a^{s-K}V_K$$

or

$$U_{s+1} = \sum_{K=1}^{(s+1)-1} a^{(s+1)-1-K}V_K,$$

so that, by induction, for any $m \geqq 1$

$$U_m = \sum_{K=1}^{m-1} a^{m-1-K}V_K. \tag{6}$$

Now suppose

$$m - 1 = kr + \alpha \qquad (0 \leqq \alpha < r, \ k \geqq 0). \tag{7}$$

Divide K in (6) by r and let the quotient be τ and the remainder σ $(0 \leqq \tau \leqq k, 0 \leqq \sigma < r)$. Set $K = \tau_1 r + \sigma_1$ $(0 \leqq \sigma \leqq \alpha)$ and $K = \tau_2 r + \sigma_2$ $(\alpha < \sigma \leqq r-1)$. Then (6) may be written

$$\begin{aligned} U_m &= \sum_{\tau_1=0}^{k} \sum_{\sigma_1=0}^{\alpha} a^{(k-\tau_1)r+\alpha-\sigma_1}V_{\tau_1 r+\sigma_1} + \sum_{\tau_2=0}^{k-1} \sum_{\sigma_2=\alpha_1}^{r-1} a^{(k-1-\tau_2)r+\alpha+r-\sigma_2}V_{\tau_2 r+\sigma_2} \\ &= \sum_{\tau_1=0}^{k} a^{(k-\tau_1)r} \sum_{\sigma_1=0}^{\alpha} a^{\alpha-\sigma_1}V_{\sigma_1} + \sum_{\tau_2=0}^{k-1} a^{(k-1-\tau_2)r} \sum_{\sigma_2=\alpha_1}^{r-1} a^{\alpha+r-\sigma_2}V_{\sigma_2} \\ &= \sum_{\tau_1=0}^{k} a^{(k-\tau_1)r} \sum_{\sigma_1=0}^{\alpha} a^{\alpha-\sigma_1}V_{\sigma_1} + \sum_{\tau_2=0}^{k-1} a^{(k-1-\tau_2)r} \sum_{\sigma_1=1}^{r-1-\alpha} a^{\alpha+\sigma_1}V_{\alpha+\sigma_1} \end{aligned}$$

on replacing in the second expression $r-\sigma_2$ by σ_1 and changing the order of summation. But

$$\sum_{\tau_1=0}^{k} a^{(k-\tau_1)r} = \frac{a^{(k+1)r}-1}{a^r-1}; \qquad \sum_{\tau_2=0}^{k-1} a^{(k-1-\tau_2)r} = \frac{a^{kr}-1}{a^r-1}.$$

Hence we obtain the following theorem.

4. THEOREM: "Let b be any integer greater than a but not a power of a, and let m be any positive integer. Let k and α denote the quotient and remainder obtained on dividing $m-1$ by the least positive integer r for which $a^{r+1}-a$ is divisible by b. Then the quotient on dividing a^m by b is given by

$$\left[\frac{a^m}{b}\right] = \frac{a^{(k+1)r}-1}{a^r-1}(a^{\alpha-1}V_1 + a^{\alpha-2}V_2 + \cdots + aV_{\alpha-1} + V_\alpha)$$

(IV)

$$+ \frac{a^{kr}-1}{a^r-1}a^\alpha(aV_{\alpha+1} + a^2V_{\alpha+2} + \cdots + a^{r-\alpha-1}V_{r-1}),$$

where $V_i = [a \cdot a_i/b]$ $(1 \leqq i \leqq r-1)$ and a_i is the least positive residue of a^i modulo b."

The numbers V_i must be found by trial. They are always less than b and can be readily computed in conjunction with the a_i's; for from (4) we have $V_s = [a \cdot a_s/b]$, $a_{s+1} \equiv a^{s+1} \equiv a \cdot a^s \equiv a \cdot a_s \pmod b$, $0 \leqq a_{s+1} < b$, or $a \cdot a_s = b \cdot V_s + a_{s+1}$, so that V_s is the quotient and a_{s+1} the remainder on dividing $a \cdot a_s$ by b. We can thus determine $V_1, V_2, \cdots, V_{r-1}$ step by step. If we have found by any means the residue system $a_1, a_2, \cdots, a_s, \cdots, a_{r-1}$, say by a table of indices modulo $b/(b, a)$, we have

$$V_s = (a \cdot a_s - a_{s+1})/b \tag{8}$$

which for small b's gives V_s by inspection.

The expressions in brackets in IV are indeed numbers in the scale a.

If the V's are assumed known, IV involves only additions and multiplications, for the first terms in each line are merely the sums of geometric progressions. Hence we have here a *formula* for the quotient of a^m by b.

There are several special cases of IV. First $\alpha = 0$ if $m-1$ is exactly divisible by r, and IV becomes

$$\left[\frac{a^m}{b}\right] = \frac{a^{kr}-1}{a^r-1}(aV_1 + a^2V_2 + \cdots + a^{r-1}V_{r-1})$$

$$= \frac{a^m-1}{a^r-1}(a^{r-1}V_{r-1} + a^{r-2}V_{r-2} + \cdots + aV_1)$$

so that we have the result:

(IVa) $$\left[\frac{a^m}{b}\right] \equiv 0 (\text{mod } a) \qquad \text{if} \quad m \equiv 0 (\text{mod } r).$$

If $b = a^r - 1$, $V_1 = V_2 = V_3 = \cdots = V_{r-2} = 0$, $V_{r-1} = 1$, and

(IVb) $$\left[\frac{a^m}{b}\right] = \frac{a^{kr} - 1}{a^r - 1} a^{r-1}.$$

If $a = 10$, the expressions in round brackets in IV become ordinary numbers with digits $V_1 \cdots V_a$; $V_{r-1} \cdots V_{a+1}$, and the geometric progressions are numbers of the form

(IVc) $$100 \cdots 0100 \cdots 0100 \cdots 01 \cdots,$$

where the 1's are separated by $r-1$ zeros.

The formulas I–IV enable us to solve any problem in division by separating a and b into sums and products in suitable ways. But IV is the only result essentially novel. Let us apply it to some numerical examples.

5. Example 1. To find the quotient when 10^{11} is divided by 13.

Here

$$a = 10, \quad b = 13, \quad m = 11, \quad a_1 = 10, \quad a \cdot a_1 = 100 = 7 \cdot 13 + 9.$$

$$\therefore v_1 = 7, \quad a_2 = 9, \quad a \cdot a_2 = 90 = 6 \cdot 13 + 12; \quad \therefore v_2 = 6, \quad a_3 = 12,$$

$$a \cdot a_3 = 120 = 9 \cdot 13 + 3;$$

$$\therefore v_3 = 9, \quad a_4 = 3, \quad a \cdot a_4 = 30 = 2 \cdot 13 + 4; \quad \therefore v_4 = 2, \quad a_5 = 4,$$

$$a \cdot a_5 = 40 = 3 \cdot 13 + 1;$$

$$\therefore v_5 = 3, \quad a_6 = 1, \quad r = 6, \quad m - 1 = 6 + 4. \quad \therefore k = 1, \quad \alpha = 4.$$

$$\left[\frac{10^{11}}{13}\right] = \left[\frac{10^{12} - 1}{10^6 - 1}\right] \cdot 7692 + \frac{10^6 - 1}{10^6 - 1} \cdot 10^4 \cdot 32$$

$$= (10^6 + 1) \cdot 7692 + 10^4 \cdot 32 = 7692307692.$$

The work here is exactly the same as in the ordinary process of short division:

$$13\overline{)100000000000}$$
$$7692307692.$$

Example 2. To find the quotient when 9^μ, where $\mu = (9^9)$, is divided by 19. Here $a = 9$, $b = 19$, $m = 9^9$, and 9, 5, 7, 6, 16, 11, 4, 17, 1 are the residues of $a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9$, so that $r = 9$.

$$m - 1 = 9^9 - 1 = (9^8 - 1)9 + 8, \quad \text{so that} \quad k = 9^8 - 1, \quad \alpha = 8.$$

Using (8), we readily find 4, 2, 3, 2, 7, 5, 1, 8 for $V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8$. Therefore if $\mu=(9^9)$ and $\nu=9^8$,

$$\left[\frac{9^\mu}{19}\right]=\left(\frac{9^\nu-1}{9^9-1}\right)(4\cdot 9^7+2\cdot 9^6+3\cdot 9^5+2\cdot 9^4+7\cdot 9^3+5\cdot 9^2+1\cdot 9+8)$$

Example 3. To find the quotient when $2^{257}-1$ is divided by 1023.

Here $a=2$, $b=a^{10}-1$, $r=10$, and IVb is applicable. Since $2^{257}\equiv 2^7 \pmod{1023}$,

$$\left[\frac{2^{257}-1}{1023}\right]=\left[\frac{2^{257}}{1023}\right]=\frac{2^{250}-1}{2^{10}-1}\cdot 2^9=2^9+2^{19}+2^{29}+\cdots+2^{249}.$$

FORMAL UNIFICATION OF GRADIENT, DIVERGENCE, AND CURL, BY MEANS OF AN INFINITESIMAL OPERATIONAL VOLUME

By VLADIMIR KARAPETOFF, Cornell University

In vector analysis, the results of the three differential operations known as taking the gradient, the divergence, and the curl of a function are radically different from each other, both from a mathematical and from a physical point of view. Nevertheless there is some formal connection among the operations themselves in that the same Hamiltonian operator "nabla" or "del" (∇) is used in all three.[1] This permits to denote the three operations as ∇, $\nabla\cdot$, and $\nabla\times$, respectively. In elementary text-books on vector analysis, the three operations and the Hamiltonian operator itself are introduced in Cartesian coördinates, thus perhaps leaving in the mind of the reader an unconscious impression that it is absolutely necessary to begin a problem by using the projections of both the operator and the operand.

On the other hand, the very purpose of vector analysis being to do away with resolving directed quantities into their components, as much as possible, a direct and unified interpretation of the foregoing differential operators in space is quite desirable. [In some advanced German works[2] this unification has been obtained by defining the gradient, the divergence, and the curl as follows:

$$\text{(A)}\qquad \nabla P=\lim_{\Delta\doteq 0}(1/\Delta v)\int_S ds P\,;\qquad \text{(B)}\qquad \nabla\cdot F=\lim_{\Delta\doteq 0}(1/\Delta v)\int_S ds\cdot F\,;$$

$$\text{(C)}\qquad \nabla\times F=\lim_{\Delta\doteq 0}(1/\Delta v)\int_S ds\times F.$$

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[1] In this article, by the Hamiltonian operator is meant the operator ∇ expressed in orthogonal coordinates, that is, $\nabla=i(\partial/\partial x)+j(\partial/\partial y)+k(\partial/\partial z)$

[2] C. Runge, *Vector Analysis* (English Translation), p. 95; W. von Ignatowsky, *Die Vektoranalysis*, vol. 1, p. 16; J. Spielrein, *Lehrbuch der Vektorrechnung*, p. 111.

In these expressions, P is a scalar point function, $\boldsymbol{F}$ is a vector point function, Δv is a small volume of any desired form, surrounding the point under consideration, S is the total surface of this volume, and $d\mathbf{s}$ is an element of this surface, thought of as a vector in the direction of the positive normal to the element. Since ∇P, $\nabla \cdot \boldsymbol{F}$ and $\nabla \times \boldsymbol{F}$ are point characteristics, the results of integration, after division by Δv, are independent of the form or the size of the chosen volume, provided that this volume and its boundary surface are in the limit reduced to zero, that is, to the point under consideration. The values of both the integrals and Δv being infinitesimally small, the process of finding ∇P, $\nabla \cdot \boldsymbol{F}$, and $\nabla \times \boldsymbol{F}$ is a generalized operation of obtaining an ordinary derivative of a function, and may be properly called *space differentiation*.

Equations (A), (B), (C) serve as general definitions of the operations under consideration. In specific applications, it is necessary to choose such a shape of the volume Δv as leads to the result in the most direct manner, preferably without actual integration and subsequent reduction of Δv to zero. It is thought that an infinitesimal volume, dv, of the shape proposed in Fig. 1 satisfies this con-

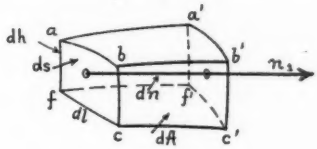

FIG. 1.

dition. It approaches a parallelopiped in shape, except that all its faces may be slightly curved, and the two opposite sides, *abcf* and *a'b'c'f'*, slightly different and skewed with respect to each other. The vector $\boldsymbol{n}_1$ is a unit normal at the center of the face *abcf*. The area *abcf* is denoted by *ds* and *cc'f'f* by *dA*. The distance between the faces *abcf* and *a'b'c'f'* is *dn*, and the length of one of the vertical edges, such as *fa*, is *dh*.

In each particular problem, the shape of the operational volume, shown in Fig. 1, is to be so chosen and the volume itself so placed in the field, as to satisfy the following two conditions:

(1) Only the faces *abcf* and *a'b'c'f'* should contribute to the integrals in the expressions (A), (B), (C);

(2) The expressions themselves should reach maximum values.

The operator ∇ which enters in equations (A), (B), (C) may then be rewritten in the form

$$\nabla = \max d_n(ds\,\boldsymbol{n}_1)/dv. \tag{1}$$

To obtain a gradient, the scalar function P, upon which this operator is to act, is to be placed in the blank space after $\boldsymbol{n}_1$. To obtain a divergence or a curl, the

vector function $\boldsymbol{F}$, preceded by a dot or a cross, is to be placed in the same blank space. The symbol d_n stands for differential and signifies a differentiation along the normal $\boldsymbol{n}_1$. Expression (1) is thus a general symbol for space differentiation corresponding to equations (A), (B), (C), provided that dv satisfies the two conditions stated above. In place of writing $d\mathbf{s}$ as a vector, it is more convenient in applications to use a scalar ds, followed by unit vector $\boldsymbol{n}_1$. For the sake of brevity we shall refer to the operator ∇, in the form of equation (1), as the *Gibbsian operator*.

It will now be explained how the shape and the position of the volume dv are to be chosen in the particular cases of gradient, divergence, and curl; and it will also be shown that the results so obtained agree with those derived from the usual elementary definitions of these operators. The advantage of definition (1), as compared with (A), (B), (C), is that it is not necessary to go through an integration and a limiting process in each particular application. The prescribed shape of the infinitesimal volume reduces the problem to a simple differentiation.

(I) *Gradient.* Let the Gibbsian operator (1) act upon a scalar point function P and let the operational volume dv be a true parallelopiped, so placed that the face $abcf$ coincides with the equi-potential surface at the point under consideration. In this case $\boldsymbol{n}_1 = \text{Const.}$; $ds = \text{Const.}$, and both may be taken outside the sign d_n. Moreover, $dv = ds \cdot dn$. Thus we have:

$$\nabla P = d_n(ds\, \boldsymbol{n}_1 P)/dv = ds\, \boldsymbol{n}_1\, d_n P/(ds\, dn) \tag{2}$$

where dn is the distance between the faces $abcf$ and $a'b'c'f'$. We thus obtain the usual expression

$$\nabla P = \boldsymbol{n}_1\, dP/dn, \tag{3}$$

where the operation (1) is performed in the direction in which the rate of change of P is a maximum.

(II) *Divergence.* Let the Gibbsian operator be provided with a "dot" after $\boldsymbol{n}_1$, and let it act upon a continuous vector function $\boldsymbol{F}$. Let the operational volume, dv, be so shaped and placed that the edges aa', bb', cc', ff', coincide with some lines of force in the immediate vicinity of the point under consideration, and the faces $abcf$ and $a'b'c'f'$ are normal to the flux. Then

$$\nabla \cdot \boldsymbol{F} = d_n(ds\, \boldsymbol{n}_1 \cdot \boldsymbol{F})/dv. \tag{4}$$

Since $\boldsymbol{n}_1 \cdot \boldsymbol{F}$ is the magnitude of the flux density along the normal, $ds(\boldsymbol{n}_1 \cdot \boldsymbol{F})$ is the total flux entering through $abcf$. Calling this flux $d\phi$, we obtain,

$$\nabla \cdot \boldsymbol{F} = [d_n(d\phi)]/dv \tag{5}$$

which is the usual definition of divergence, viz., excess flux per unit volume.

(III) *Curl.* Let the Gibbsian operator (1) be provided with a "cross" after $\mathbf{n}_1$ and let it act upon a continuous vector function $\boldsymbol{F}$. A small portion of the flux near the point under consideration is shown in Fig. 2 by the horizontal and vertical projections of lines of force, such as pq and rs. The osculating plane at the point w of pq is horizontal, so that the vertical projection of pq is a very nearly horizontal straight line. The element fc of pq is supposed to be parallel to the axis XY.

Of all the lines of force adjacent to pq, and distant from it by the same amount dn, that one is chosen (rs) for which the "twist" is the greatest, that is, the one whose osculating plane at x forms the greatest angle with the osculating plane of pq at w.

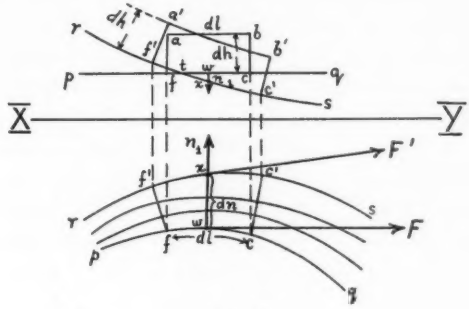

FIG. 2.

The operational volume is constructed as follows: The edge fc (Fig. 1) is placed on pq, the edge $f'c'$ on rs; the faces $bb'c'c$ and $aa'f'f$ are made normal to the lines of force. The average height, dh, of the face $abcf$ is taken to be equal to that of $a'b'c'f'$ (up to infinitesimals of the second order). The face $a'b'c'f'$ is skewed with respect to $abcf$ about the axis $\mathbf{n}_1$, in accordance with the above mentioned twist between the elements cf and $c'f'$ of the two lines of force. Both faces are normal to the respective osculating planes. We then have:

$$\nabla \times \boldsymbol{F} = d_n(ds\, \mathbf{n}_1 \times \boldsymbol{F})/dv \tag{6}$$

The area ds may be replaced by the product $dh\ dl$, and the volume dv by the product $dh\ dA$. Factoring out and cancelling the constant quantity dh in the numerator and denominator, we get

$$\nabla \times \boldsymbol{F} = d_n(\mathbf{n}_1 \times \boldsymbol{F}\, dl)/dA \tag{7}$$

where

$$dA = \text{area } cff'c'.$$

$\boldsymbol{Fdl}$ is the line integral of F along fc; and because it forms a vector product with $\boldsymbol{n}_1$, this line integral is to be thought of as a vector normal to the osculating plane of pq. The symbol d_n means that a geometric difference is to be formed between this line integral and the corresponding value $\boldsymbol{n}_1' \times \boldsymbol{F}'dl'$, along rs. The sum of the line integrals of F along cc' and ff' being an infinitesimal of a higher order, the numerator of equation (7) represents, in the limit, the total line integral of the vector function $\boldsymbol{F}$ along the contour $cc'f'f$. The ratio of this integral to the area, dA, of the contour, as in equation (7), is the curl of $\boldsymbol{F}$, according to the usual definition.

Let the flux be two-dimensional (co-planar); in other words, let there be no skewing of rs with respect to pq, both lines of force being curves in the same plane. Then both partial line integrals, $\boldsymbol{n}_1 \times \boldsymbol{F}dl$ and $\boldsymbol{n}_1' \times \boldsymbol{F}'dl'$, are normal to the plane of the curves, and consequently the curl is perpendicular to the planes of the flux lines. Skewing the flux, that is, making it three-dimensional, adds a component of the curl in the direction of the flux itself. Since rs is chosen in the direction of maximum skewing, expression (7) contains not only the whole component of the curl normal to the plane of pq, but the whole component along $\boldsymbol{F}$ as well; in other words, it gives the whole curl vector.

The foregoing investigation supplements the author's earlier article on the *Divergence and curl in a vector field, in terms of curvature and tortuosity*, Philosophical Magazine, vol. 31 (1916), p. 528.

EXPRESSIONS FOR THE GENERAL DETERMINANT IN TERMS OF ITS PRINCIPAL MINORS

By E. B. STOUFFER, University of Kansas

It was shown by MacMahon[1] in 1893 and later in a more simple manner by Muir[2] that only n^2-n+1 of the 2^n-1 principal minors (including the determinant itself) of the general determinant of the nth order are independent. Thus, for $n=1, 2, 3$ the principal minors are all independent but for $n>3$ there exist 2^n-n^2+n-2 relations among them.

In 1924 the author of the present paper gave a method[3] of expressing the general determinant of order $n>3$ in terms of n^2-n+1 of its principal minors of order less than n. Several expressions for the general determinant of order four in terms of its principal minors of order three or less had already been obtained[4]

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[1] Philosophical Transactions of the Royal Society of London, vol. 185, pp. 111–160.

[2] Philosophical Magazine and Journal of Science, (5), vol. 38, pp. 537–541.

[3] Transactions of American Mathematical Society, vol. 26, pp. 356–368.

[4] See, e. g., MacMahon, loc. cit.; Nansen, Philosophical Magazine and Journal of Science, (5) vol. 44, pp. 362–67; Muir, Transactions of the Royal Society of Edinburgh, vol. 39, pp. 323–339.

but previous efforts to solve the problem in a general way had apparently been stopped by an erroneous theorem by MacMahon.[1] This theorem states that a general determinant of odd order cannot be expressed in terms of its principal minors of lower order. The error in MacMahon's proof arises from the assumption that certain relations among the principal minors are all independent. Such is not the case. It should be added that MacMahon's apparent proof that there are only two independent relations involving the principal minors of a general determinant of even order is also incorrect. The source of the error is the same.

It is the purpose of the present paper to give a method of expressing the general determinant of any order in terms of only fourteen of its principal minors of lower order. The expressions obtained are much simpler than those given by the previous method.

Let A represent a determinant of the nth order in which the element in row i and column j is denoted by a_{ij}. Furthermore, let $(A_{klmn}\cdots)$ denote the principal minor of A obtained by striking out all rows and columns except those numbered $k, l, m, n, \cdots$.

Muir[2] has shown that the values of the principal minors of a determinant A with general elements a_{ij} are not affected if certain $n-1$ elements are each put equal to unity. One such set consists of the elements $a_{1j}(j>1)$ and we shall accordingly assume in the remainder of this paper that those elements are equal to unity.

The following theorem[3] is important for our future considerations:

The principal minors

$$(1)\qquad (A_i),\ (A_{ij}),\ (A_{1ij}),\quad i,j=1,2,\cdots,n;\quad i<j,$$

constitute a total of n^2-n+1 *independent minors.*

We shall actually use this theorem only for $n=4$. For this case the direct proof by means of the functional determinant is quite simple.

For $n=4$ with $a_{12}=a_{13}=a_{14}=1$ we have the following results:

$$(2)\qquad \begin{aligned} a_{ii} &= (A_i) \quad (i=1,2,3,4);\\ a_{j1} &= (A_1)(A_j)-(A_{1j}) \quad (j=2,3,4);\\ a_{ij}a_{ji} &= (A_i)(A_j)-(A_{ij}) \quad (i,j>1;\ i\neq j);\\ a_{j1}a_{ij}+a_{i1}a_{ji} &= (A_{1ij})-(A_1)(A_{ij})+2(A_1)(A_i)(A_j)\\ &\quad -(A_j)(A_{1i})-(A_i)(A_{1j}),\quad (i,j>1;\ i\neq j). \end{aligned}$$

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[1] MacMahon, loc. cit.

[2] Muir, Philosophical Magazine and Journal of Science, (5), vol. 38, pp. 537–541.

[3] Stouffer, loc. cit., p. 357.

Thus, for the case $n=4$, the expressions for all the elements of A in terms of the minors (1) follow immediately from these equations. The substitution of these expressions for the elements of A gives an expression for A itself and for its one remaining principal minor (A_{234}) in terms of thirteen of its principal minors of order three or less.[1]

The process outlined above works for determinants of any higher order. It has the advantage of involving principal minors of order no higher than three but the expression for A becomes very complex as the number n^2-n+1 of minors involved increases.

We shall now show that it is possible to go directly from the expression for the determinant of order four in terms of thirteen of its principal minors to an expression for the determinant of any order in terms of fourteen of its principal minors. For this purpose we make use of the so-called law of extensible minors which has been stated by Muir[2] as follows: *If any identical relation be established between a number of the minors of a determinant or between the determinant itself and a number of its minors, the determinants being denoted by means of their principal diagonals, then a new theorem is always obtainable by merely choosing a line of letters with new suffixes and annexing it to the end of the diagonal of every determinant, including those of order zero, occurring in the identity.*

It is obvious that the "extension" of a principal minor of one determinant is always a principal minor of the new determinant. Consequently, by "extending" each principal minor in the expression for the determinant of order four we can obtain an expression for a determinant of any desired order in terms of its principal minors. The necessity of keeping the expression homogeneous requires the introduction of the principal minor of maximum order consisting of new elements only. We thus have proved our main theorem:

Expressions for the general determinant of order $n>4$ in terms of fourteen of its principal minors follow immediately from the expressions for the determinant of order four in terms of thirteen of its minors.

The expressions thus obtained involve one principal minor of order $n-4$, four of order $n-3$, six of order $n-2$, and three of order $n-1$.

It is obvious that other expressions for the general determinant in terms of its principal minors may be obtained by replacing the principal minors of order ofur or greater involved in the above expressions by their equivalents in terms

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[1] It is interesting to note that there are in general eight distinct determinants of order four whose principal minors (1) have the same values. This becomes evident when we observe that equations (2) contain three pairs of simultaneous equations, each pair consisting of a linear and a quadratic equation. Similarly, there are eight minors (A_{234}) determined by a given set of minors (1). The elements of A are uniquely determined in general when in addition to the values of (1) there is known a particular one of the eight associated values of (A_{234}).

[2] Muir, Transactions of the Royal Society of Edinburgh, vol. 30, pp. 1–4.

of their principal minors of lower order. A continuation of the process gives expressions for the original determinant in terms of its principal minors of order three or less. Of course, not more than 2^n-n^2+n-2 of the relations obtained in this way can be independent.

It is obvious from the well-known relations among the minors of a determinant and its adjoint that another relation involving only the determinant and its principal minors follows immediately from any one of the expressions obtained as indicated above if we replace each principal minor which appears by its cofactor, provided the expression is again made homogeneous by means of the determinant itself.

A THEOREM ON ALGEBRAIC QUADRATIC FORMS AND ITS APPLICATION IN THE GENERAL THEORY OF RELATIVITY

By P. Y. CHOU, California Institute of Technology

In the general theory of relativity the value of $\cos\theta$, where

$$\cos\theta = g_{\alpha\beta}\frac{dx_1^\alpha}{ds_1}\frac{dx_2^\beta}{ds_2}, \qquad ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta, \tag{1}$$

and θ is defined to be the angle between the tangents of the two curves s_1 and s_2 intersecting at a point (x^1, x^2, x^3, x^4) in the 4-dimensional space S_4, is ambiguous. The like holds for the extremals obtained from

$$\delta\int ds = 0. \tag{2}$$

These extremals may be either maxima or minima. The indeterminateness arises in each case from the fact that the fundamental quadratic form ds^2 is indefinite. It is usually proved in books on Riemannian Geometry[1] that $\cos\theta$ defined in (1) is less than unity provided the fundamental quadratic form ds^2 is positive definite. However, under the relativity hypothesis that any infinitely small space-time continuum can be always reduced to a Galilean space, we can show that $|\cos\theta| \geqq 1$ and that the geodesics of (2) are maximizing lines in S_4. In order to substantiate these two statements we have to establish the following theorem:

If for the real indefinite quadratic form $a_{\alpha\beta}x_\alpha x_\beta$ of rank n, signature $2-n$,

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[1] L. P. Eisenhart, *Riemannian Geometry*, p. 37.

there are two sets of values of $x_i (i=1, \cdots, n)$, y_i *and* z_i *such that* $a_{\alpha\beta}y_\alpha y_\beta = A_1^2 > 0$, $a_{\alpha\beta}z_\alpha z_\beta = A_2^2 > 0$, *then the polar form,* $a_{\alpha\beta}y_\alpha z_\beta$, *satisfies the condition,*

$$|a_{\alpha\beta}y_\alpha z_\beta| \geqq |A_1 A_2|. \tag{3}$$

Since $a_{\alpha\beta}x_\alpha x_\beta$ has rank n and signature $2-n$, by a real non-singular linear transformation of x_i,

$$x_i' = c_{i\alpha}x_\alpha,$$

where the determinant

$$|c_{ik}| \neq 0, \tag{4}$$

we obtain[1]

$$a_{\alpha\beta}x_\alpha x_\beta = x_1'^2 - x_2'^2 - x_3'^2 - \cdots - x_n'^2.$$

Therefore

$$a_{\alpha\beta}y_\alpha y_\beta = y_1'^2 - y_2'^2 - \cdots - y_n'^2 = A_1^2;$$

$$a_{\alpha\beta}z_\alpha z_\beta = z_1'^2 - z_2'^2 - \cdots - z_n'^2 = A_2^2.$$

Unless y_i and z_i are all zero, which case we exclude, y_1' and z_1' can not vanish. Hence

$$1 = \frac{A_1^2}{y_1'^2} + \frac{y_2'^2}{y_1'^2} + \cdots + \frac{y_n'^2}{y_1'^2}; \qquad 1 = \frac{A_2^2}{z_1'^2} + \frac{z_2'^2}{z_1'^2} + \cdots + \frac{z_n'^2}{z_1'^2}. \tag{5}$$

We are dealing with real quantities, so the following relation is evident:

$$\left\{\frac{|A_1|}{|y_1'|} - \frac{|A_2|}{|z_1'|}\right\}^2 + \left\{\frac{|y_2'|}{|y_1'|} - \frac{|z_2'|}{|z_1'|}\right\}^2 + \cdots + \left\{\frac{|y_n'|}{|y_1'|} - \frac{|z_n'|}{|z_1'|}\right\}^2 \geqq 0. \tag{6}$$

By (5), (6) becomes

$$1 - \frac{1}{|y_1'z_1'|}[|A_1A_2| + |y_2'z_2'| + |y_3'z_3'| + \cdots + |y_n'z_n'|] \geqq 0,$$

and therefore

$$|y_1'z_1' - (y_2'z_2' + \cdots + y_n'z_n')| \geqq |y_1'z_1'| - |y_2'z_2'| - \cdots - |y_n'z_n'| \geqq |A_1A_2|. \tag{7}$$

By the inverse transformation from x_i' to x_i on the left hand side of (7), we obtain our final result,[2]

$$|y_1'z_1' - (y_2'z_2' + \cdots + y_n'z_n')| = |a_{\alpha\beta}y_\alpha z_\beta| \geqq |A_1A_2|. \tag{8}$$

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[1] M. Bôcher, *Introduction to Higher Algebra*, p. 148.

[2] Ibid., p. 128.

In the definition (1) of $\cos\theta$, the rank of the quadratic form $g_{\alpha\beta}(dx^\alpha/ds)(dx^\beta/ds)$ is 4 and the signature is -2; and A_1 and A_2 are both unity. By (8), $|\cos\theta| \geqq 1$. Hence in order that θ should be real we may adopt the following definition of the angle θ between two intersecting curves in S_4,

$$\cosh\theta = \pm\, g_{\alpha\beta}(dx_1^\alpha/ds_1)(dx_2^\beta/ds_2).$$

It follows that for two intersecting time-like world lines, i.e., world lines for which $ds^2>0$, the condition of orthogonality can never be satisfied.

The Weierstrass' E-function for the integral

$$(9)\qquad I = \int_{t_1}^{t_2} f\left(x, \frac{dx}{dt}\right)dt, \qquad f\left(x, \frac{dx}{dt}\right) = \left\{g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\right\}^{1/2} > 0,$$

where for convenience's sake we write x, dx/dt for x^i, dx^i/dt $(i=1, \cdots, 4)$, respectively, in the argument of f, and where t is the independent parameter which can eventually be put equal to s, defined by $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$, is given by

$$(10)\qquad E(x,\dot{x},\dot{y}) = f(x,\dot{y}) - f(x,\dot{x}) - (\dot{y}^\delta - \dot{x}^\delta)\frac{\partial}{\partial\dot{x}^\delta}f(x,\dot{x})$$

$$= -\frac{1}{f(x,\dot{x})}(\dot{y}^\delta - \dot{x}^\delta)g_{\delta\beta}\dot{x}^\beta = -\frac{1}{f(x,\dot{x})}(g_{\delta\beta}\dot{x}^\beta\dot{y}^\delta - 1),$$

where $\dot{x}^i = dx^i/ds$ and the direction, $\dot{y}^i$, is sufficiently close to the direction $(\dot{x}^1, \cdots, \dot{x}^4) \neq (0, \cdots, 0)$. A necessary condition for the extremal

$$(11)\qquad x^i = x^i(t), \quad (i = 1, \cdots, 4); \quad x_1^i = x_1^i(t_1), \quad x_2^i = x_2^i(t_2)$$

of (9) to be a maximizing line is[1]

$$(12)\qquad E(x,\dot{x},\dot{y}) \leqq 0.$$

By a slight modification the above condition also insures that (11) be a relatively strong or relatively weak maximum.[2] As the theorem stands in its present form in (3) it is not directly applicable to (10). Hence a modified proof of the theorem for the present purpose is necessary.

Replace $|y_i'|$ and $|z_i'|$ in (6) by y_i' and z_i' respectively. By (5) we obtain

$$(13)\qquad 1 - \frac{1}{y_1' z_1'}[\,|A_1A_2| + y_2'z_2' + y_3'z_3' + \cdots + y_n'z_n'\,] \geqq 0.$$

The transformation (4) is one-to-one and $x_1' = c_{1\alpha}x_\alpha$ is a continuous function of x_i $(i=1, \cdots, n)$. Moreover, $x_1' \neq 0$, unless $x_i = 0$ $(i=1, \cdots, n)$. Hence for a set of values of x_i, z_i, which are in the closed neighborhood of the set

[1] G. A. Bliss, *Calculus of Variations*, p. 138. Replace the word "minimum" by "maximum."

[2] Ibid., p. 157.

$(y_1, \cdots, y_n) \neq (0, \cdots, 0)$ the signs of y_1' and z_1' can not be different because otherwise if we vary x_i from y_i to z_i $(i=1, \cdots, n)$, x_1' must vanish at least once, which is impossible. Consequently we may multiply (13) throughout by $y_1'z_1'$ and get

$$y_1'z_1' - (y_2'z_2' + y_3'z_3' + \cdots + y_n'z_n') \geqq |A_1A_2|.$$

By the inverse transformation on this, we obtain $a_{\alpha\beta}y_\alpha z_\beta \geqq |A_1A_2|$.

The above result is directly applicable to (10) in which both A_1 and A_2 are unity. Consequently Weierstrass' necessary condition (12) is established for the integral (9).

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. NOTE ON STEREOGRAPHIC PROJECTION

By THEODORE BENNETT, University of Illinois

Two fundamental theorems concerning stereographic projection state that (1) any circle on the stereographic sphere is projected into a circle on the stereographic plane, and conversely; and (2) the angle of intersection of two curves on the sphere is equal to the angle of intersection of their stereographic projections. The writer has examined a number of texts which treat this subject, and in all of them these two theorems are proved analytically. It is not difficult to prove them without algebraic assistance, using only the standard facts concerning planes and spheres which one learns in a first course in solid geometry; but such a proof does not seem particularly elegant. It is the purpose of this note to prove these theorems by the methods of synthetic projective geometry. The interest is therefore in the method used rather than the results established.

Since the theorems are of a metric character we shall first consider the relationship between projective and Euclidean geometry. Euclidean geometry in the plane is a specialization of projective geometry characterized by a special fixed line (the line at infinity) and two fixed points on that line. These fixed points, I and J, are called the circular points at infinity. Every circle in the plane passes through I and J, and every conic which passes through these points is a circle. The lines CI and CJ, which join I and J to a finite point C,

are called isotropic lines. Notwithstanding the fact that these lines are imaginary, they play a fundamental role in Euclidean geometry; and they are of particular importance when we are specializing a theorem of projective geometry to obtain a corresponding theorem of Euclidean geometry. These lines may be used to find the angle between two given lines. If the given lines are CA and CB, they form with CI and CJ a pencil which has a definite cross ratio $(C-ABIJ)$; then

$$\angle ACB = \tfrac{1}{2}i \log (C - ABIJ), \quad \text{where } i = \sqrt{(-1)}.$$

This formula, which is called the formula of Laguerre,[1] will be used later.

Euclidean geometry of three dimensions is a specialization of projective geometry characterized by a special fixed plane (the plane at infinity) and a conic in that plane which is called the absolute[2] or sphero-circle at infinity. Every sphere passes through the absolute, and every quadric which contains it is a sphere. Any finite plane meets the absolute in two points which are the circular points at infinity in that plane.

Finally, let us recall a few facts concerning quadric surfaces. Any non-singular quadric has two sets of rectilinear generators; we call each set a regulus. No two lines of the same regulus intersect, but any line of one regulus meets every line of the other. Through any point of the quadric there is one line of each regulus. The hyperboloid of one sheet and the hyperbolic paraboloid have real generators; other types of non-singular quadrics have imaginary generators. Any plane section of a quadric is a conic; if the plane is tangent to the quadric the conic degenerates into the two generators which pass through the point of tangency. In particular, the generators of a sphere are isotropic lines, and any tangent plane cuts the sphere in the two isotropic lines (in the tangent plane) which pass through the point of tangency. Two quadrics intersect in a twisted curve of order four, which sometimes degenerates into two or more curves of lower order. For example, if two quadrics have two straight lines in common the residual intersection is a conic, which must be a plane section.

Let us now consider the stereographic sphere, with P and S two diametrically opposite points. From P we project Q, any point on the sphere, into Q', on the plane Σ, which touches the sphere at S. If Π is the plane tangent to the sphere at P, we know that Π and Σ have a common line at infinity, and therefore they have the same circular points, I and J, in which their common line at infinity meets the absolute.

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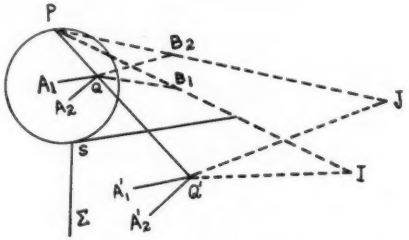
[1] Pascal, *Repertorium*, vol. 2, p. 28.

[2] Snyder and Sisam, *Analytic Geometry of Space*, p. 52.

Any circle on the sphere is cut out by a plane, p, which we assume does not pass through P. The lines PI and PJ lie entirely on the sphere, since they are the generators through P. Therefore the points in which they pierce the plane p are two points of the circle in which p meets the sphere. Moreover, when these points are projected upon Σ (from P) they fall at I and J. Therefore the projection of the circle is a conic (on Σ) passing through the circular points I and J, whence it is a circle.

Conversely, if P is joined to every point of an arbitrary circle in Σ, we have a quadric cone, K, which has PI and PJ for two of its generators, since the circle must pass through I and J. But PI and PJ lie on the sphere, and this line pair forms part of the intersection of K with the sphere. The remaining part is a conic, which must be a plane section of the sphere. Since all plane sections of a sphere are circles, we see that any circle on Σ is the projection of a circle on the sphere.



If the plane p passes through P, the projection of the circle cut out by p is composite. The points of the circle other than P are projected into the points of the straight line in which p and Σ intersect. The point P is projected into the whole line at infinity in Σ. Since the line at infinity contains the circular points, this projection forms a degenerate circle. This completes the proof of the first theorem stated at the beginning of this note.

It may be well to state here that a single finite line and a circle are not equivalent from the standpoint of projective geometry, although they are equivalent from the standpoint of inversive geometry (the geometry associated with the function theory in the complex plane). The reason is that in one case the infinite region of the plane is defined to be a line, and in the other case a single point.

We now prove the second fundamental theorem. Let the curves on the sphere meet at Q, with tangents QA_1 and QA_2. Since a projection sends a curve and one of its tangent lines into a curve and a corresponding tangent line, the curves which meet at Q are sent into two curves which meet at Q', with tangents $Q'A_1'$ and $Q'A_2'$, where $Q'A_1'$ is the projection (from P) of QA_1, and similarly for $Q'A_2'$. The plane QA_1A_2 touches the sphere at Q, and therefore cuts out

the two generators through Q, namely the isotropic lines QB_1 and QB_2. The isotropic lines PI and PJ are the generators through P, and of these two lines one meets QB_1 and the other meets QB_2. Therefore the line pencils $(Q-A_1A_2B_1B_2)$ and $(Q'-A_1'A_2'IJ)$ are projective, whence their cross ratios are equal. But in the plane QA_1A_2 the lines QB_1 and QB_2 are the two isotropic lines through Q, and similarly, in Σ, $Q'I$ and $Q'J$ are the two isotropic lines through Q'. Therefore, from the formula of Laguerre, the angles A_1QA_2 and $A_1'Q'A_2'$ are proportional to the logarithms of the cross ratios of the corresponding line pencils. Hence the equality of the angles follows from the equality of the cross ratios.

II. Some Geometric Illustrations for the Elementary Course in Differential Equations

By J. D. Tamarkin, Brown University

1. The equation linear in y and y':

$$\frac{dy}{dx} + p(x)y + q(x) = 0. \tag{1}$$

The family of integral curves of (1) possesses a curious property as follows:

(*A*) *Draw the line $x=\xi$ and at the points of intersection of this line with the integral curves of (1) draw tangents to all integral curves. All these tangents either pass through the same point (which depends on ξ), or they are parallel.*

The proof of (*A*) is based upon another property of the integral curves of (1), which is well known:

(*B*) *If we draw any two particular integral curves (C_1), (C_2) of (1), then the general integral curve of (1) is the locus of points which divide in a constant ratio the segments of vertical lines ($x=\xi$; ξ is a parameter) intercepted by (C_1) and (C_2).*

This property follows immediately from the fact that if y_1 and y_2 are any two particular solutions of (1), then the general solution y of (1) is given by

$$y = y_1 + C(y_2 - y_1),$$

where C is an arbitrary constant.

Now to prove (*A*), draw any three integral curves of (1)

$$(C_1)\ y = y_1(x)\ ;\quad (C_2)\ y = y_2(x)\ ;\quad (C_3)\ y = y_3(x).$$

Let P_i, P_i' $(i=1, 2, 3)$ be the points of intersection of (C_i) with the vertical lines $x=\xi$ and $x=\xi'$ respectively. It follows from (*B*) that

$$\overline{P_1P_2}/\overline{P_2P_3} = \overline{P_1'P_2'}/\overline{P_2'P_3'}\ ;$$

hence the chords $P_i P_i'$ $(i=1, 2, 3)$ either are concurrent or else they are parallel. If we make $\xi' \to \xi$, the chord $P_i P_i'$ approaches its limiting position, that is the tangent to (C_i) at P_i. Hence the tangents to (C_i) $(i=1, 2, 3)$ at the points P_i are either concurrent or else parallel, which proves (A).

It is readily seen that the tangents at the points P_i are parallel when and only when the slopes of the integral curves of (1) for $x=\xi$ do not depend on the ordinate y of the curve, that is when the coefficient $p(\xi)$ of y in (1) is zero for this particular value $x=\xi$.

Denote by $T_\xi(x, y)$ the point of concurrency of the tangents to the integral curves of (1) at their points of intersection with the vertical $x=\xi$. As ξ varies the point T_ξ describes a certain locus which may be of great use for the graphical integration of (1). To obtain the equation of the locus of T_ξ, write the equation of the tangent to the integral curve of (1) at the point (ξ, y):

$$Y - y = -[p(\xi)y + q(\xi)](X - \xi).$$

Substitute any two particular values of y, as for instance, $y=0$ and $y=1$, and solve the resulting equations with respect to X and Y. This gives

$$X = \xi + [1/p(\xi)]; \quad Y = -\frac{q(\xi)}{p(\xi)}.$$

2. The equation linear in x and y:

$$y = x\phi(y') + \psi(y'). \tag{2}$$

If we are given any differential equation of the first order

$$F(x, y, y') = 0, \tag{3}$$

the notion of the "curves of equal slope" of (3) is very important in the discussion of the geometric properties of the integral curves. The curve of equal slope (equal y_0') is the locus of the points at which the tangents to the integral curves have the same slope y_0'.

The family of these curves of equal slope is obtained immediately from (3) if we consider y' as a parameter which has a fixed value for any particular curve of the family.

In the case of equation (2), linear in x and y, the curves of equal slope are straight lines. The tangents to all the integral curves of (2) drawn at their points of intersection with any line

$$y = x\phi(y_0') + \psi(y_0')$$

are parallel (with the slope y_0').

Now, the integral curves of (2) have the property analogous to (*B*), the only difference being that the "segments of the vertical lines" should be replaced by the "segments of the lines of equal slope."

(*C*) *If we draw any two particular integral curves* (C_1), (C_2) *of* (2), *then the general integral curve of* (2) *is the locus of points which divide in a constant ratio the segments of the lines of equal slope intercepted by* (C_1) *and* (C_2).

To prove (*C*) we have to remember that equation (2) "is integrated by differentiation." Take differentials and observe that $dy = y'dx$:

$$y'dx = dx\phi(y') + x\phi'(y')dy' + \psi(y')dy'. \tag{4}$$

This is a linear equation in x and dx/dy', which can be solved by quadratures, giving x as a function of y' and of an arbitrary constant, C. On substituting in (2) we get y in terms of y' and C. If we consider y' as a parameter we obtain, for each fixed value of C, a particular integral curve of (2). Take any two particular integral curves (C_1) and (C_2) which correspond to any two particular values C_1 and C_2 of C. Let

$$x = x_i(y'), \quad y = y_i(y') = x_i\phi(y') + \psi(y'),$$

be the resulting equations of (C_i) $(i=1, 2)$. Equation (4), being linear in x and dx/dy', has the general solution $x = x_1 + C(x_2 - x_1)$. Accordingly the equations of the general integral curve of (2) become

$$x = x_1 + C(x_2 - x_1)\ ; \quad y = y_1 + C(y_2 - y_1)$$

which shows that the segment P_1P_2 joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is divided by the point $P(x, y)$ in a constant ratio independent of y'. Thus (*C*) is proved.[1]

III. The Origin of the Term "Root" in Chinese Mathematics

By C. C. Ma, Columbia University

In the *Shih Wu Chi Yuan*, a historical work written by Kao Cheng in the year 1448, the invention of writing, in the proper sense of the term, is attributed to Tsang Chieh who is said to have lived in the twenty-eighth century, B.C. Tradition states that he devised his characters by imitating the footprints of birds, which is quite as reasonable a theory as several that have been ad-

[1] For many years I used to give the properties (*A*) and (*C*) in the elementary course in differential equations. As far as I am aware, these properties are not found in any of the standard textbooks on differential equations, except in the recent book by E. L. Ince, *Ordinary Differential Equations* (Longmans, Green Co., London, 1927), which contains an incomplete formulation of property (*A*); the case where the tangents are parallel is omitted (p. 15, problem 9).

vanced concerning the origin of Western alphabets.[1] Whether or not the Chinese theory is a mere fable, the fact remains that the better established history of the development of such characters as the one representing a mathematical root is filled with interest. The older ideograms have been studied with much care by Fu Ying Kei in his work entitled *Liu Shu Fen Lei*, published in 1765. In this he gives the early forms of the word which we translate as "root" as follows:

In modern Chinese works the word appears in several forms, among them the following:

It is pronounced *kun* and means the lower part of a plant, that part which fixes itself in the earth and draws nutriment from the soil. Thus we speak of the roots of grass, or the root of a tree, and hence metaphorically of the bottom of any object; of the root of a mountain; of the root, origin, cause of anything; of the root or source of virtue; of an ancestor (the root of a family tree); of the fundamental note of a chord; or of the root of bitterness, error, sin, or evil. In our mathematical works it always refers to square root or cube root,—that is, to the basic number from which we obtain special products (squares or cubes).

Although the form of the ideogram has changed from time to time, the fundamental meaning of the word *kun* has remained the same. It is because of this fact that the writer was surprised at reading, in Dr. Gandz's interesting article *On the origin of the term root*,[2] the statement, "The Chinese, indeed, do use the word *kun* to mean root, grass, and shrub, and the Hindus also use the word *mula* for the root of a plant, but *this is very likely due to the Arabic influence*, which is so often seen in China and which may have spread into India by way of China."

It is true that in the records of our Tang Dynasty (618–907) there begins to appear the evidence of Arabic influence, but before that period we had fully thirty centuries of culture. In this extended period we developed our own type of mathematics and our own terminology. The word *kun* was familiar to thousands of our scholars and appears in many of our early books, including the Book of Rites, the Book of Poetry, the Book of History and others equally well-known in the East. Not only does it appear, but it has always the same meanings, and hence it is historically impossible that we should have derived it from the relatively late and transitory civilization of the Arabs.[3]

[1] On a new theory of the origin of all alphabets, see I. W. D. Hackh, *The history of the alphabet*, Scientific Monthly, August, 1927.

[2] In this Monthly, vol. 33 (1926), pp. 261–265.

[3] The author wishes to express his thanks to Professor David Eugene Smith for assistance in preparing this note for publication.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED

CURTIS, D. R. and MOULTON, E. J. *Trigonometry, Plane and Spherical.* New York, D. C. Heath and Company, 1927. xi+264 pages. $2.40.

FINE, H. B. *Calculus.* New York, The Macmillan Company, 1927. viii+421 pages.

HASTINGS, C. S. *New Methods in Geometrical Optics.* New York, The Macmillan Company, 1927. vii+103 pages.

JEFFREYS, HAROLD. *Operational Methods in Mathematical Physics.* (Cambridge Tracts, Number 23) Cambridge, University Press, 1927. vii+101 pages.

LIE, S. *Gesammelte Abhandlungen.* Sechster Band. Leipzig, B. G. Teubner and Oslo, H. Aschehoug and Company, 1927. xxiv+940 pages.

LINDOW, M. *Numerische Infinitesmalrechnung.* Berlin, Ferd. Dummlers Verlag, 1927. viii+176 pages. 15 marks.

MAIR, D. B. *Fourfold Geometry.* New York, D. Van Nostrand, 1927. viii+183 page. $2.75.

NAIK, V. B. and BONDALE, S. B. *Elementary Solid Geometry.* Poona, Aryabhushan Press, 1926. v+220 pages.

ROOT, R. E. *The Mathematics of Engineering.* Baltimore, The Williams and Wilkins Company, 1927. xiii+540 pages. $7.50.

RUNNING, T. R. *Graphical Mathematics.* New York, John Wiley and Sons, 1927. viii+91 pages. $1.75.

SPEISER, A. *Theorie der Gruppen von endlicher Ordnung.* Zweite Auflage. Berlin, Julius Springer, 1927. ix+251 pages. 15 marks.

VEBLEN, OSWALD. *Invariants of Quadratic Differential Forms.* (Cambridge Tracts, Number 24) Cambridge, University Press, 1927. viii+102 pages.

WELLS, W. and HART, W. W. *Modern Plane and Solid Geometry.* New York, D. C. Heath and Company, 1927. x+480 pages. $1.64.

REVIEWS

Le Calcul des Probabilités, son Évolution Mathématique et Philosophique. By L. G. DU PASQUIER. Paris, J. Hermann, 1926. xxi+304 pp.

The nine chapter headings with the principal topics discussed in each are: *Historical Introduction.* The three currents in the evolution of probability. The invention of the calculus of probability. The mathematical current. The sociological and stocastical currents. The philosophical current. *Elements of the Calculus of Probability.* The classical definition of mathematical probability. Fundamental theorems. Combinatory analysis. Stirling's formula. *The Series of Repeated Trials.* The empirical law of chance or the law of great numbers. Various concepts of discrepancy. Repeated trials and Newton's binomial theorem. The first part of Bernoulli's theorem. *Bernoulli's Theorem and the Theory of Discrepancies.* The mathematical probability that the fluctuation of a series of repeated trials takes a fixed value. The mathematical probability that the discrepancy of a series of repeated trials lies between fixed limits. Standard deviation. Equiprobable limits. Formulae of chance. Precision. Dispersion. Mean value and mathematical expectation. *Various Interpretations of the Calculus of Probability.* The psychological, practical, logical, empirical,

and inductive interpretations. The principle of zones of comportment. *Mathematical Probability Based on the Theory of Assemblages. The Logical Definition of Probability.* Mathematical probability and the theory of assemblages. Paradoxes of the calculus of probability. Mathematical probability and ignorance. Mathematical probability and the principle of zones of comportment. Mathematical probability and statistical regularity. *Applications to Physics.* The introduction of the calculus of probability in physical chemistry. Review of the kinetic theory of gases. Distribution of molecular velocities. The law of Maxwell-Boltzmann. Reversible phenomena and the principle of Carnot. Statistical determinism and fluctuations. Other applications to physics. *Mathematical Probability Based on the Theory of Aggregates. The Stocastical Definition of Probability.* The empirical aggregate. The mathematical aggregate. A new definition of mathematical probability. Four simple operations for deducing new mathematical aggregates. *Mathematical Probability and Experience.* Two fundamental postulates and experience. The postulate of fortuity. Logical systems and experience. The appendix contains two tables of values for the probability function.

This detailed analysis shows us several new topics in the subject. The author has a viewpoint which is more mathematical than that of Keynes but more philosophical than those of Arne Fisher, Coolidge, Borel, Poincaré, or Bachelier.

Stocastics is defined to be the science which applies the theorems of the calculus of probability to numbers collected by statistics in such a way as to show the existences of real causes whose action combines with that of fortuitous causes (p. 23). A more formal definition is that given in chapter VIII, which also illustrates the use of other terms peculiar to the author. "Definition 5.—An assemblage of objects or things, which are included in a general notion and which are arranged in the order of a measurable or numerically expressible character, forms an aggregate or *syllepte.* It is characterized by its *law of distribution, frequency function,* or *diadose.* The study of these aggregates from the point of view of the calculus of probabilities constitutes stocastics."

F. A. FORAKER

College Algebra. By W. B. FORD. Revised edition. New York, The Macmillan Co., 1926. vii+288 pages.

This book follows the lead of most college algebras which have appeared in the last few years, in that it has included more review material and has excluded some of the more advanced topics, such as partial fractions and limits and series. Also most of the exercises of an involved character have been replaced by simpler ones. Some teachers will probably have a feeling that such a procedure is not entirely justified.

The first forty-two pages are essentially a review. This is followed by the usual topics included in such a text. The following are a few features of the development of these topics. The chapter on "Variation" is treated more fully than usual and a large number of problems from physics is inserted. The derivative is defined and used to find maximum and minimum values of a function. With this definition at hand one wonders why Newton's method of approximation was not included in the chapter on the theory of equations. On the other hand, most readers will be pleased to find Sylvester's method of elimination and the definition of eliminant included in the chapter on determinants.

The book is well written. The explanations are clear, and all principles are illustrated by examples carefully worked out. One ambiguity should be noted. On pages six and forty-eight the word "cancel" is used. On page six it means divide and on page forty-eight it means subtract. This double meaning of the word is confusing to the student. Hence it might be better to replace the word "cancel" by the word which suggests the process involved.

Very few typographical errors occur, and the general make-up of the pages is good.

J. H. WEAVER

Charts and Graphs. By K. G. KARSTEN, New York, Prentice-Hall, Inc., 1925. xl+734 pages. Price $6.00.

Graphical methods have been found extremely useful for at least three general purposes: (1) the correlation and simplification of vast amounts of historical, economic, experimental or any other kind of data, (2) the use of these data to predict development beyond present knowledge, (3) the preparation in graphical form of any kind of information so that it may be more easily understood by the general public or any special group of people.

The increasing use and universal appeal of graphical methods create a need for a book describing the evolution, the uses and abuses of all kinds of graphs. This need has been met by the author, who sets as his object the telling of the whole story of charts and graphs in the simplest words at his command. The result is a book of 734 pages with 498 illustrations, well indexed and written in an interesting, non-technical style. The small amount of necessary mathematics has been thoroughly explained so that it can be understood without special training. The general headings are:

I. Non-mathematical Charts (4 chapters); II. Amount-of-change Analysis (26 chapters); III. Rate-of-change Analysis (8 chapters); IV. Special Analyses (such as curves of error, probability curves and curve fitting—6 chapters); V. Calculating Charts (such as curves for formulae, slide rules and alignment charts or nomographs—5 chapters); VI. Two- and Three-dimension Data

(such as models, surfaces and relief maps—7 chapters); VII. Conclusion (statistical materials and function of charts—2 chapters).

In reading this book the reviewer was favorably impressed with the wealth of illustration covering every type of chart, graph, and diagram and also with the complete description of the construction and use that accompanies each illustration.

The author has provided a liberally illustrated textbook for those whose business it is to collect, study, and disperse statistical information. At the same time he has prepared a reference book for persons, such as business executives, engineers, and experimentalists, who can use to their own advantage a greater variety of charts and graphs than they do at present.

HUBERT H. RACE

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

THE NAPERIAN CLUB OF DEPAUW UNIVERSITY, Greencastle, Indiana.

The officers for the year 1926–1927 were: President, William Nusbaum, '27; vice-president, Ruth Quebbeman, '27; secretary, Josephine Rode, '27; treasurer, Dever Colson.

The following programs were given at the regular monthly meetings:

November 11, 1926. "The life and work of John Napier" by Boyd Crawley, '27.

January 13, 1927. "Theorem of the nine point circle" by Charles La Hue, '28.

February 8. "Number Systems" by Professor Greenleaf. "Geometric fallacies" by Ruth Quebbeman, '27.

March 10. "Acoustics in buildings" by Dever Colson, '27.

April 21. "The teaching of mathematics for the last three quarters of the present century" by Helen Huhn, '28.

On April 28 and 29 the Indiana section meeting of the Mathematical Association of America was held at Depauw University. The Naperian Club was very much interested in helping obtain Professor Kunz of the University of Illinois as one of the speakers.

(Report by Josephine Rode)

THE NEWTONIAN SOCIETY OF THE STATE COLLEGE OF WASHINGTON, Pullman, Washington.

President, Miss Florence Johnson; secretary, Miss Arlene Perry; reporter, Mrs. Opal Priest.

November 3, 1926. Prof. C. A. Isaacs—"Numbers".

November 17. Mr. Schulkenoff—"Null spaces."

December 15. Dorothy Webster—"Transfinite numbers." Arlene Perry—"Zeno's paradoxes."

December 29. Mrs. Mills—"Euclidean and Riemannian geometry."

February 10, 1927. Mr. Calogaris—"A recent trip to Greece."

February 24. Miss Swannack—"Elliptic geometry." Mr. Schulkenoff—"Mathematical worlds."

March 3. Miss Verna Betz—"History of determinants." Prof. Isaacs—"Some determinants connected with the triangle and the tetrahedron."

March 10. Miss Davis—"Evolution of the solar system." Mr. Calogaris—"Laplacian law of densities."

March 17. Miss Pell—"Solving a cubic equation." Prof. Irwin—"Approximation curves."

March 29. Miss Freeman—"Approximation curves." Prof. Colpitts—"The students' loan association."
April 14. Prof. Hutchinson—"The application of calculus to physics."

(Report by Professor C. A. Isaacs)

The Mathematics Club of Wellesley College, Wellesley, Massachusetts.

The Mathematics Club of Wellesley College made some interesting studies in the year 1926–1927:

October 22, 1926. Discussion and review by members of books available on the Mathematics Browsing Shelf in the College Library.

November 19. Discussion of topics for which the year 1926 might be considered an anniversary: The work of Eratosthenes, Edmund Gunter, Albert Girard, Wilbar Snell. One member spoke on mathematics as it was first introduced into colleges.

January 7, 1927. Meeting in the treasure room of the College Library. Prof. Lennie P. Copeland pointed out certain unusual details in the old mathematical volumes kept there.

February 25. "Problems of antiquity and their solution:" (1) "Geometric representation of the roots" by Elizabeth Mitchell; (2) "Trisection of an angle" by Frances Baume; (3) "Duplication of a cube" by Elizabeth Peek; (4) "Spirals" by Doris Miller.

March 11. "The Geometry of imaginaries" by Dr. William C. Graustein of Harvard University.

May 10. The department of mathematics entertained the club.

May 28. Club dinner and election of officers for year 1927–1928: President, Leona Bailey, '28; vice president, Elizabeth Peek, '28; treasurer, Mary Parsons, '28; secretary, Doris Raine, '29; Junior executive, Esther Kirkbride, '29; faculty advisor, Prof. Marion E. Stark.

(Report by Frances A. Hartman)

The Gordon N. Armstrong Mathematics Club of Ohio Wesleyan University, Delaware, Ohio.

The program of the club for the year 1926–1927 was as follows:

October 13, 1926. "Irrational numbers by Prof. Rowland.

November 10. "Maclaurin's theorem" by Mr. Foote.

December 8. Social meeting.

January 19, 1927. "The Circle" by Mr. McCullough.

On April 16, 1927 the Beta Chapter of Ohio of Pi Mu Epsilon was installed at Ohio Wesleyan University. Meetings are held bi-monthly and one meeting each month is held in conjunction with the Gordon N. Armstrong Mathematics Club.

The Pi Mu Epsilon program for the remainder of the year was as follows:

April 27, 1927. "Integration by imaginary substitution" by Miss Leonard.

May 11. "Tricks in mathematics" by Prof. Crane.

May 25. "The relation of mathematics to psychology" by Mr. Valentine.

(Report by Edward T. Prosser)

Pi Mu Epsilon, University of Kentucky, Lexington, Kentucky.

A chapter of Pi Mu Epsilon was established at the University of Kentucky this year with the following officers: Director, Dean P. P. Boyd; vice-director, Professor J. Morton Davis; secretary, Professor F. E. LeStourgeon; treasurer, Mr. E. J. Canaday; librarian, Professor E. L. Rees. The White Mathematics Club was continued with Dean P. P. Boyd as president, and Professor F. E. LeStourgeon as secretary. Meetings of the two organizations alternated during the year. The combined program was as follows:

December 2, 1926. "Trisection of an angle" by Professor E. L. Rees.

January 20, 1927. "Double tangents of quartics" by Dean P. P. Boyd.

February 10. "Line, surface, and space integrals" by D. E. South, Instructor.

February 24. "Groups of substitutions" by Professor F. E. LeStourgeon.

March 10. "Elliptic functions" by Mr. D. O. Streyffeler, Instructor. "Application of mathematics to architecture" by Mr. E. J. Canaday, Instructor.
March 24. "Sir Isaac Newton" by Mr. W. W. Chambers, Instructor. "Certain mathematical equations in the theory of gases" by Mr. Grant Willey, '27.
April 28. "Coupled oscillations" by Mr. R. B. Scott, Graduate. "The path of an α-particle through an atom of a heavy metal" by Mr. Madison Cawein, Graduate.
May 30. "Pedal curves" by Mr. M. C. Brown, Instructor.

(Report by Professor F. E. LeStourgeon)

The Mathematics Club of Cooper Union, New York City.

The officers for the year 1926–1927 were: Edwin T. Schwarz, '27, president; Henry Wilhelm, '27, vice-president; Fred Mertens, '29, secretary-treasurer. Meetings were held at intervals of two or three weeks throughout the year, with the following program:
November 9, 1926. "Applications of probability to automatic telephony" by William H. Spahn of the Bell Telephone Laboratories.
December 1. "Tricks on the slide-rule" by Edwin T. Schwarz, '27.
December 22. "Integral right triangles" by A. H. Beiler of the Interboro Rapid Transit Company.
January 12, 1927. "Unicursal curves" by Henry Wilhelm, '27.
January 26. "Summation of algebraic series" by Samuel Lubkin, '27.
February 16. "π by chance" by Henry Eckhardt, '27.
March 9. "Mental calculations, with an exposition of methods" by Abraham Rosenbaum, '28.
March 26. Joint meeting of the Cooper Union Student branches of the A. S. M. E., A. I. E. E., A. S. C. E., the Mathematics Club, and the Chemistry Club. "How to become a successful engineer" by Farley Osgood, past president of the A. I. E. E.
April 6. "Paradoxes of infinity" by Walter Prenowitz, Instructor.
April 20. "A Photographic representation of a crinkly curve" by Edwin T. Schwarz, '27. At this meeting the following officers were elected for the coming year: Peter Douglas, '28, president; James McConaghy, '30, vice-president; Edward Trapani, '29, secretary-treasurer.

(Report by Professor H. W. Reddick)

The Undergraduate Mathematics Club of the University of Iowa, Iowa City, Iowa.

The officers of the Undergraduate Mathematics Club of Iowa University for the year 1926–27 were: Faculty advisor, Professor Roscoe Woods; president, Mr. C. R. Wilson, graduate student; secretary-treasurer, Miss Eva E. Latta, '28.

The following programs were given at the club meetings during the year:
November 4, 1926. "Paper folding" by Professor R. P. Baker.
November 18. "The nine-point circle" by Miss Ruth Balluff, M. S.
December 2. "Number systems" by Mr. A. H. Blue, M. S.
January 13, 1927. "Logarithmic graphs" by Mr. C. S. Carlson, Graduate.
February 3. "Algebra of points" by Mr. C. A. Messick, M. S.
February 17. "Hyperbolic functions" by Mr. W. M. Davis, Graduate.
March 3. "The slide rule" by Mr. Harry Rice, Graduate.
March 31. "Some unusual numbers" by Miss Phebe Williams, '28.
May 5. "Measurement of distances in astronomy" by Mr. Walter Long, '29.
May 19, 1927. "Transcendentalism of e" by Dr. N. B. Conkwright.

At the last meeting, the following officers were elected for the year 1927–28: Faculty advisor, Dr. N. B. Conkwright; president, Mr. H. A. Meyer, graduate; secretary-treasurer, Miss Dorothy McCoy, graduate.

The club meets on Thursday afternoons at four o'clock; tea is served each time before the beginning of the program.

(Report by Eva E. Latta)

The Mathematics Club of Miami University, Miami, Ohio.

November, 1926. "The attraction of spherical shells" by Mr. MacDougal. "Prime numbers" by Mrs. Reed.

December. "Squaring the circle" by Mr. DeLong. "DeMoivre's theorem" by Miss Hughes.

January, 1927. "The trisection of an angle" by Miss Runyon. "Curves used in surveying" by Mr. Halbedel. "The theory of the planimeter" by Mr. Armstrong.

February. "The Einstein theory" by Mr. Rumbaugh.

March. "The Pythagorean theorem" by Miss Young. "A graphical solution of the cubic equation" by Miss Williams. "Arithmetic progressions and interpolations" by Mr. Bulow.

April. "Central forces" by Miss Fisher. "The general number" by Miss Treichler. "Geometric interpretation of an imaginary angle" by Mr. Serviss.

May. "Logarithms of negative numbers" by Miss Zwick. "Nomography" by Mr. Tischer.

All of the talks were given by students of the various classes under the direction of the mathematics professors. Its main purpose was to bring to the students an opportunity of acquiring a knowledge of topics rarely discussed in the classroom. (Report by Harold F. Sielaff)

The Harvard Mathematical Club, Harvard University, Cambridge, Mass.

The officers for 1926–27 were: Mr. Robin Robinson, president; Mr. Morris Marden, secretary. Officers for 1927–28 are: Mr. Clarence I. Lubin, president; Mr. Griffith B. Price, secretary.

Meetings for 1926–27

October 20, 1926. "Riemann's theory of algebraic functions and functions related to them" by Professor W. F. Osgood.

November 3. "Some properties of skewsquares" by Mr. S. B. Sommerville.

November 17. "Some topics in the calculus of variations" by Mr. T. L. Smith.

December 1. "European mathematics of today" by Professor G. D. Birkhoff.

December 15. "Legendre's normal forms and some applications" by Mr. T. R. Long.

January 5, 1927. "Motor calculus" by Mr. C. I. Lubin.

January 19. "Multiple integrals with singular integrands dependent on a parameter" by Mr. A. B. Brown.

February 16. "Some mathematical ideas in electrical engineering" by Dr. Joseph Slepian, Research Consulting Engineer, Westinghouse Electric and Manufacturing Company.

March 2. "An application of theta functions in the theory of numbers" by Mr. C. N. Liu.

March 16. "A recent treatment of maxima and minima" by Mr. S. S. Cairns.

March 30. "The determination of analytic functions" by Professor Philip Franklin, Massachusetts Institute of Technology.

April 13. "The parallelism of Levi-Civita" by Mr. H. B. Hammatt.

April 27. "Some circles related to the triangle" by Mr. G. B. Price.

May 11. "Some properties of entire functions" by Mr. F. C. Jonah.

May 25. "Parallelism in Finsler space" by Dr. Harry Levy.

(Report by Robin Robinson)

Sever Mathematical Club of Harvard University, Cambridge, Mass.

The officers for 1926–27 were: Mr. Stuart B. Sommerville, president; Mr. Edward H. Cutler, secretary.

The officers for 1927–28 are: Mr. John J. Hinrichsen, president; Mr. Joseph K. Peterson, secretary.

Meetings for 1926–27

October 27, 1926. "The stability of floating bodies" by Mr. Robin Robinson.

November 10. "The structure of the world" by Mr. D. D. Kosambi.

December 8. "Diophantine analysis" by Mr. S. S. Cairns.

January 12, 1927. "Some solutions of the cubic and the biquadratic" by Mr. J. J. Hinrichsen.

February 9. "Sir Isaac Newton's pasturage problem" by Mr. R. M. Barclay.
February 23. "Some mathematical scandals" by Mr. Charles Wexler.
March 9. "Point sets" by Mr. W. S. Seidel.
March 28. "The four-color map problem" by Miss M. E. Steinmetz.
April 11. "The lateral area of an oblique cone with circular base" by Mr. C. W. T. Bunnell.
May 4. "Recurring series" by Mr. E. C. Berkeley.
May 18. "On the roots of polynomials" by Mr. Morris Marden.

(Report by S. B. Sommerville)

The Junior Mathematics Club of the University of Chicago, Chicago, Illinois

The officers for the year 1926–1927 were: R. G. Archibald, president; F. R. Bamforth, chairman of committee on program; May M. Beenken, chairman of committee on arrangements; A. O. Hickson, secretary and treasurer.

The officers for the year 1927–1928 are: B. W. Jones, president; May M. Beenken, chairman of committee on program; Ruth Mason, secretary and treasurer.

The program for the year 1926–1927 was as follows:
October 20, 1926. "Some interesting topics in algebra" by Miss L. W. Griffiths.
November 3. "Pythagorean theorems and Pythagorean triads" by Dr. J. S. Georges.
November 17. "The four color problem" by Dr. Orrin Frinck.
December 8. "Projective equivalents of trigonometric functions" by Miss M. M. Beenken.
January 5, 1927. "Introduction to the structure of algebras" by Mr. A. A. Albert.
January 19. "On some questions regarding non-euclidean geometries" by Miss Louise Lange, Ph. D.
February 2. "Popular probability problems" by Mr. C. E. Rhodes.
February 16. "Mathematical card tricks" by Mr. B. W. Jones.
March 2. "Magic squares" by Mr. R. A. Hefner.
March 30. "Some notes concerning the theory of relativity" by Miss Louise Lange, Ph. D.
April 9. Social evening at the home of Professor H. E. Slaught.
April 13. "The algebraic method of obtaining the derivative" by Mr. H. H. Downing.
April 27. "On the congruences of lines generated by a projective relationship between two planes" by Miss L. Hoffman, Ph. D.
May 11. "A theorem of Bertrand" by Mr. Lincoln LaPaz.
May 25. "Hyperbolic functions" by Mr. C. J. Rees.

(Report by Professor H. E. Slaught)

The Mathematics Club of Hunter College, New York City.

The officers for 1926–27 were Miss Lena Kalan, president; Miss Antoinette Finocchi, secretary; Miss Ruth Chatwin, treasurer; Dr. Roger Johnson, advisor. The following meetings were held during the year:
October 5, 1926. Business meeting.
October 17. Hike and picnic.
October 19. Party for freshmen.
October 26. "Sturm's theorem" by Professor Tomlinson Fort.
November 4. "The numbers of Fibonacci" by Dr. Roger Johnson.
November 14. "Projective ornament" by Mr. Claude Bragdon.
December 3. "Infinite series" by Professor W. B. Fite of Columbia University.
February 15, 1927. Business meeting.
March 8. Party.
March 15. "The cycloid" by Jean Hutchinson.
March 29. "Magic squares" by Josephine Stephenson.
April 5. "Comets" by Lewis D. Hill, Professor of Physics.
April 19. "The conchoid" by Lillian Glass.

(Report by Miss Finocchi)

The Mathematics Club of McKendree College, Lebanon, Illinois.

The Mathematics Club of McKendree College, Lebanon, Illinois, was organized in October, 1926, with twelve charter members. Students who finish the first semester of freshman mathematics with a good record are eligible to associate membership. Students who enroll for sophomore calculus are eligible to active membership. Meetings are held monthly. The programs for the school year 1926–27 were as follows:

November 4, 1926. Social meeting.

December 8. Topic: "Invariants of the general equation of the second degree in two variables." Papers by Mr. V. R. W. Hardy, Mr. Lorin Mitchell, Mr. Emery Martin, and Professor C. J. Stowell.

January 7, 1927. Topic: "The teaching of high school mathematics." Papers by Miss Emma Bergmann, Miss Verdie Correll, Mr. Eugene Smith and Professor C. E. Vick.

March 3. "Analytic geometry before Descartes" by James Stuart. "The contribution of Descartes to analytic geometry" by Mr. Clifton Gould. "The early development of calculus" by Mr. Charles Jack.

April 14. "Elements of the theory of probability" by Mr. Lee Baker. "The method of least squares" by Mr. J. Wendell Dunn.

May 10. Social meeting.

(Report by Professor Stowell)

The Mathematics Club of Hood College, Frederick, Maryland.

The officers for 1926–27 were: President, Elizabeth Slifer; vice President, Anna Lee Schaidt; Secretary, Grave Lough; Reporter, Dorothy Kerper.

October 12. 1926. Initiation of new members.

November 9. "A mathematical melange" by Dr. Apple.

December 14. "Blaise Pascal" by Elva Kurtz. "Anecdotes about famous mathematicians" by Ruth Anderson, Dorothy Dougherty, Katherine Kehm, Dorothy Kerper, Ethel Landgraff, Ethel Sauble.

January 11, 1927. "Trend of mathematics in secondary schools" by Prof. Remsberg.

February 15. "Development of the calendar:" "Early calendars" by Anna Lee Schaidt, "Julian and Augustan" by Irene Speicher, "New calendar" by Ethel Dallmeyer.

March 18. "Binary scales of notation" by Susan Hirsch. "Cardan's rings" by Katherine Woodfill.

April 12. "Map making" by Dorothy Bobb. "Newton" by Gladys Allison, Grace Lough. "Euclid" by Nellie Burrows. Election of officers.

April 28. Picnic.

May 10. "Mathematics and music" by Miss Wiggin.

At each meeting magazine reports from mathematical magazines found in the library were given by different members of the club.

(Report by Miss Lough)

The Mathematical Society of Rutgers University, New Brunswick, New Jersey.

The Mathematical Society at Rutgers University held regular monthly meetings during the year 1926–1927 at which the following papers were presented by members of the faculty and by the student members:

Prof. W. V. N. Garretson—"A new representation of the complex number."

Prof. E. P. Starke—"Applications of the complex variable."

Prof. S. E. Brasefield—"Vectors."

Prof. Richard Morris—"Positive integral solutions of an indeterminate equation."

Prof. C. M. Huber—"Non-euclidean geometry."

Mr. A. E. Meder, Jr.—"Mathematical analysis of statistics."

Mr. Carl Gronquist—"Analysis of a structure by the method of least work."

Mr. H. M. Walter—"Actuarial mathematics."

Messrs. Veatch and Wittes—"Partial fractions."
Mr. Klaessig—"Stereographic projections."
Messrs. Eason and Worden—"Euler's theorem on homogeneous functions."
Mr. Clayton—"The Jacobian a covariant."

(Report by Professor Morris)

The Mathematical Society of the New Jersey College for Women, New Brunswick, New Jersey.

During the year 1926–1927, the Mathematical Society of the New Jersey College for Women was featured by a number of social events in addition to the regular monthly meetings.

The Society sent one of the student members as a delegate to the meetings of the Association in Philadelphia.

The social events consisted of a tea for the new members, a christmas party, a dinner in honor of Professors Titsworth and Morris, a subscription dance, a farewell tea to the seniors.

The Society designated and adopted a pin for members.

During the month of March, a symposium on the teaching of mathematics was held once each week.

The following papers and talks were given at the regular meetings:
Prof. A. A. Titsworth—"Reminiscences of the work on the geodetic survey of New Jersey."
Prof. C. M. Huber—"An account of non-euclidean geometries."
Mr. A. E. Meder, Jr.—"Graphical methods for computation."
Prof. Richard Morris—"Positive integral solutions of an indeterminate equation."
Miss Ruth Thompson—"Mathematics in investments."
Prof. E. P. Starke—"Tri-linear co-ordinates."
Miss Elsie Lundt gave a summary of her trip to the meetings of the Association at Philadelphia.
Miss Blackford—"The Slide Rule."
Miss Van Etten—"Fallacies."
Miss Spinetti—"Mathematics and philosophy."
Miss Bilderback—"A sketch of Isaac Newton."
Miss Bump—"Fluctions."
Miss Robinson—"Notes on 'Principia'."

A roll call was held at which every member responded with some mathematical problem or interesting mathematical fact.

(Report by Professor Morris)

Mu Theta Epsilon, University of California, Berkeley, Calif.

The officers for the year 1926–1927 were: Henrietta Sommer, president; E. B. Roessler, vice-president; Lucye E. Morris, secretary; Dean Smith, treasurer.

The program for the year consisted of the following:

August 25, 1926. "Perfect numbers" by D. H. Lehmer.

September 22. "What is mathematics?" by Professor Bernstein.

October 20. "The determination of time" by C. H. Smiley. "Seismology" by Henrietta Sommer.

December 1. "Generalization of measurement of distances and angles" by Beryl Britton. "The derivation of the secant column formula" by E. B. Roessler.

January 19, 1927. "Algebraic numbers" by D. C. Duncan.

February 5. Annual initiation and banquet.

February 16. "Geometrical inversion" by Helena Kusick. "The Naperian base" by D. H. Lehmer.

March 10. "Simplification of an equation in astronomical work" by H. Miles. "Primality" by Emma Trotskaia.

April 14. "The rate of reaction" by Jerome Martin.

(Report by Miss Irma Wicht)

THE IRRATIONAL CLUB OF THE UNIVERSITY OF WYOMING, Laramie, Wyoming.

The following is the program of the Irrational Club at the University of Wyoming for the year 1926–27:

November 9, 1926. Organization meeting at the home of Professor and Mrs. O. H. Rechard. Election of officers and social meeting. Officers for the year: Positive square root, Stella Lavergne; negative square root, Harry Cole; keeper of the log and bones, Reiva Niles.
November 23. "Magic squares" by Fredia Conner.
December 7. "Determinates" by Phillip Pepoon.
December 20. "What we see in the heavens" by Greta Neubauer.
January 20, 1927. "Einstein and his work" by O. H. Rechard.
February 3. "An evening with the telescope" by C. F. Barr.
February 17. Social meeting with program of mathematical stunts.
March 3. "Non-euclidean geometry" by Marion Linnville.
March 17. Historical topics by various members of the club.
April 12. "Squaring the circle"—Discussion by several members.
May 5. Club Picnic—Final meeting.

(Report by Reiva Niles, Keeper of the Log and Bones)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3302. Proposed by R. E. Gaines, University of Richmond.

Determine the envelope of the chord of a central conic which subtends a right angle at a given focus.

3303. Proposed by Otto Dunkel, Washington University.

Show that the sequence $C_0, C_1, C_2, C_3, \cdots$, defines the arithmetic nth root of N, where

$$C_{i+1} = C_i \frac{(n+1)N + (n-1)C_i^n}{(n-1)N + (n+1)C_i^n},$$

n is a positive integer, and N and C_0 are any positive numbers. See this MONTHLY [1927, 131]. Simpler sequences for this purpose are indicated on pages 367–368.

3304. Proposed by J. V. Uspensky.

The Eulerian numbers $E_1, E_2, E_3, \cdots$, are defined by the development

$$\sec x = 1 + E_1\frac{x^2}{1\cdot 2} + E_2\frac{x^4}{1\cdot 2\cdot 3\cdot 4} + E_3\frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots,$$

whence it follows that $E_1=1$, $E_2=5$, $E_3=61, \cdots$. Let p be a prime number $=4m+1$. Denoting by f an *odd* number between 0 and p which satisfies the congruence $f^2+1\equiv 0 \pmod{p}$, show that

$$E_1 + E_3 + \cdots + E_{2m-1} \equiv \frac{1-(-1)^\mu}{2} \pmod{p}, \quad \text{where } \mu = \frac{f-1}{2}.$$

$$1 + E_2 + E_4 + \cdots + E_{2m} \equiv \frac{1-(-1)^\mu}{2} \pmod{p}, \quad \text{where } \mu = \frac{f-1}{2}.$$

For example, for $p=17$, we have $m=4$ and $f=1$, and

$$E_1 = 1 \equiv 1; \qquad E_3 = 61 \equiv -7; \qquad E_5 = 50521 \equiv -3; \qquad E_7 = 199360981 \equiv 9$$

$$E_1 + E_3 + E_5 + E_7 \equiv 0 \pmod{17}.$$

UNSOLVED PROBLEMS

Solutions are desired for the following unsolved problems proposed from January 1, 1913 to December 31, 1917. The number of each problem is printed in bold face type, with the year and page numbers following. Some of these problems were reprinted as indicated by the second reference. The numbers in parentheses are the incorrectly printed numbers. A list of unsolved problems proposed after December 31, 1917 will be printed later.

Algebra. **461,** 1916, 209 and 1919, 414.

Geometry. **446,** 1914, 191; **470,** 1915, 228 and 1919, 414; **472,** 1915, 267; **477,** 1915, 337; **478,** 1915, 338; **494,** 1916, 210; **510,** 1917, 231; **523,** 1917, 426.

Note. 478 is the same problem as 523.

Calculus. **348,** 1913, 312 and 1919, 268; **353,** 1914, 55 and 1919, 312; **415,** 1916, 301; **429,** 1917, 231; **432,** 1917, 287; **434,** 1917, 328 and 1918, 119; **436,** 1917, 388.

Mechanics. **279** (274), 1913, 222 and 1919, 214; **287,** 1914, 55 and 1919, 312; **291,** 1914, 122; **299,** 1914, 267; **308,** 1915, 268; **309,** 1915, 162; **313,** 1915, 202; **315,** 1915, 309; **343,** 1917, 124; **344,** 1917, 177; **351,** 1917, 328.

Number Theory. **191** (187), 1913, 196 and 1919, 214; **196,** (192), 1913, 223 and 1919, 214; **202,** 1913, 313 and 1919, 312; **205,** 1914, 55; **232,** 1915, 202; **234,** 1915, 268; **238,** 1916, 19; **263,** 1917, 171; **266,** 1917, 288; **270,** 1917, 389; **272,** 1917, 427; **274,** 1917, 467; **275,** 1917, 467.

SOLUTIONS

3228 [1926, 525]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The perpendiculars dropped from the vertices of a triangle upon the lines joining the mid-points of the opposite sides to the orthocenter of the triangle meet the respective sides of the triangle in three collinear points.

The line joining these three points is perpendicular to the Euler line of the triangle.

I. Solution by T. C. Esty, Amherst College

Take the origin, O, at the orthocenter of the triangle ABC and let α, β, γ be the position vectors of the vertices A, B, C respectively. Then $\alpha \cdot (\beta - \gamma) = \beta \cdot (\gamma - \alpha) = \gamma \cdot (\alpha - \beta) = 0$, whence

$$\alpha \cdot \beta = \beta \cdot \gamma = \gamma \cdot \alpha. \tag{1}$$

The position vectors of the mid-points A', B', C' of the sides BC, CA, AB are, respectively $\frac{1}{2}(\beta+\gamma)$, $\frac{1}{2}(\gamma+\alpha)$, $\frac{1}{2}(\alpha+\beta)$. The equation of BC is $\rho = t\beta + (1-t)\gamma$. If AL is perpendicular to OA', its equation is $(\rho-\alpha)\cdot(\beta+\gamma)=0$. Hence the point of intersection, L, of BC and AL is given by the last two equations, with

$$t = (\beta+\gamma)\cdot(\alpha-\gamma)/(\beta^2-\gamma^2) = (\gamma\cdot\alpha-\gamma^2)/(\beta^2-\gamma^2) = (\alpha\cdot\beta-\gamma^2)/(\beta^2-\gamma^2).$$

Hence the position vector of L is $\lambda=[(\alpha\cdot\beta-\gamma^2)\beta+(\beta^2-\alpha\cdot\beta)\gamma]/(\beta^2-\gamma^2)$. By a cyclic permutation of the letters α, β, γ and the use of equation (1), we find the position vectors of the corresponding points M, N on the sides CA, AB to be

$$\mu=\frac{[(\alpha\cdot\beta-\alpha^2)\gamma+(\gamma^2-\alpha\cdot\beta)\alpha]}{(\gamma^2-\alpha^2)}; \qquad \nu=\frac{[(\alpha\cdot\beta-\beta^2)\alpha+(\alpha^2-\alpha\cdot\beta)\beta]}{(\alpha^2-\beta^2)}.$$

A necessary and sufficient condition for the collinearity of three points whose position vectors are λ, μ, ν is the existence of scalars x, y, z not all zero such that

$$x\lambda+y\mu+z\nu=0 \quad \text{and} \quad x+y+z=0.$$

Since $$(\alpha\cdot\beta-\alpha^2)(\beta^2-\gamma^2)\lambda+(\alpha\cdot\beta-\beta^2)(\gamma^2-\alpha^2)\mu+(\alpha\cdot\beta-\gamma^2)(\alpha^2-\beta^2)\nu=0$$

and $$(\alpha\cdot\beta-\alpha^2)(\beta^2-\gamma^2)+(\alpha\cdot\beta-\beta^2)(\gamma^2-\alpha^2)+(\alpha\cdot\beta-\gamma^2)(\alpha^2-\beta^2)=0,$$

it follows that L, M, and N are collinear.

To prove that the line LMN is perpendicular to the Euler line, we note that a vector parallel to the former is given by

$$\mu-\lambda=(\gamma^2-\alpha\cdot\beta)(\beta^2-\gamma^2)^{-1}(\gamma^2-\alpha^2)^{-1}[(\beta^2-\gamma^2)\alpha+(\gamma^2-\alpha^2)\beta+(\alpha^2-\beta^2)\gamma].$$

Since the position vector of the centroid is $\frac{1}{3}(\alpha+\beta+\gamma)$, we can prove our proposition by showing that $(\alpha+\beta+\gamma)\cdot(\mu-\lambda)=0$, which follows from the fact that

$$(\alpha+\beta+\gamma)\cdot[(\beta^2-\gamma^2)\alpha+(\gamma^2-\alpha^2)\beta+(\alpha^2-\beta^2)\gamma]=0.$$

II. Solution by Otto Dunkel, Washington University

Let H be the intersection of the altitudes AA_h, BB_h, CC_h of the triangle ABC; and let A_hB_h cut AB in C_1; B_hC_h cut BC in A_1; and C_hA_h cut CA in B_1. It will be shown that these are the three points of the problem and that they are collinear. The points H, A_h, C, B_h determine a complete quadrilateral and from it we see that A, C_h, B, C_1 are harmonic. Similarly, A, B_h, C, B_1 are harmonic. The harmonic pencil $A_1(A, C_h, B, C_1)$ cuts AC in A, B_h, C and a fourth point which must be B_1. Hence A_1, B_1, C_1 lie in a straight line. Through H draw HK parallel to AB and HJ meeting CC_1 perpendicularly at J. Then $H(B_h, K, A_h, J)$ has its rays perpendicular, respectively, to those of $C(A, C_h, B, C_1)$; it is therefore harmonic and it cuts AB in B, ∞, A, C'. Therefore C' is the middle point of AB. A similar proof applies for A' and B'.

We now prove that $A_1B_1C_1$ is the radical axis of the nine-point circle (N) and the circumcircle (S) with the centers N and S, respectively. It will then follow that NS, the Euler line, is perpendicular to $A_1B_1C_1$.

Let E be the middle point of HC; produce CS and JHC' until they meet in C_2. Now SC' is equal and parallel to CE, and it follows that $SC_2=CS$. Hence J and C_2 lie on (S). A circle (E) with center E passes through C, J, H, B_h, A_h. Thus (E) and (S) have the common chord CJ; also (E) and (N) have the common chord B_hA_h. Therefore the radical axis of (S) and (N) passes through C_1; similarly, it passes through A_1 and B_1, and the proof is complete.

Also solved by H. E. Arnold, C. F. Barr, J. W. Clawson, Max Coral, H. A. DoBell, and B. C. Patterson.

3230 [1926, 525]. Proposed by C. N. Schmall, New York City.

Determine the point in an ellipse at which the two focal distances include the greatest angle.

Solution by T. L. Smith, Harvard University

Let P be a point on the first quadrant of the ellipse with the foci F' and F. Let the bisector of the angle P of the triangle $F'FP$ cut $F'F$ in Q. Set $\angle QPF=\theta$, $\angle FQP=\psi$, $F'P=x$, $FP=y$, $FF'=2c$. Then $x+y=2a$ and hence $QF=cy/a$. The law of sines applied to the triangle FQP gives $\sin\theta=c\sin\psi/a$, where θ is acute. Hence $\psi=90°$ gives the maximum $\theta=\sin^{-1}(c/a)$.

Also solved by R. P. AGNEW, J. M. BARBOUR, A. G. CLARK, H. A. DOBELL, J. B. MEYER, A. PELLETIER, J. B. REYNOLDS, and F. W. WINTERS.

3232 [1927, 45]. Proposed by J. Rosenbaum, Milford, Conn.

Prove that in a tetrahedron, if three of the face angles at one vertex are equal, each less than or equal to a right angle, then any of the angles of the face opposite that vertex is less than the supplement of one of the three equal angles.

SOLUTION BY THE PROPOSER

Denote the tetrahedron by $ABCD$, and let D be the common vertex of the three equal angles, each angle being equal to α. Also denote the sides of the triangle ABC by a, b, and c, in the usual manner; and the edges DA, DB, and DC by x, y, and z respectively.

If α is a right angle then, $a^2=y^2+z^2$, $b^2=z^2+x^2$, $c^2=x^2+y^2$; and hence $a^2<b^2+c^2$.

Therefore, angle A, that is any angle, of triangle ABC is acute, and hence less than the supplement of α.

For the case when α is acute, we denote $\cos\alpha$ by k, and prove that for any angle such as A, of triangle ABC, the opposite side, a, satisfies the inequality $a^2<b^2+c^2+2kbc$, thus:

From triangles DCA, DAB, $x-z<b$, $y-x<c$.

Since these inequalities hold also when their left hand members are replaced by their absolute values, we have a right to multiply these inequalities obtaining

$$-yz < x^2 - zx - xy + bc.$$

Multiply by the positive quantity $2k$,

$$-2kyz < 2kx^2 - 2kzx - 2kxy + 2kbc.$$

Replacing the positive term $2kx^2$ of the right hand member by the greater quantity $2x^2$, and adding y^2+z^2 to each side of the resulting inequality, we obtain

$$y^2 + z^2 - 2kyz < z^2 + x^2 - 2kzx + x^2 + y^2 - 2kxy + 2kbc.$$

From the triangles DBC, DCA, and DAB, this reduces to the inequality which we desired to prove; and it completes the proof.

Also solved by J. B. REYNOLDS.

3234 [1927, 45]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The median through a given vertex of a triangle meets the Apollonian circle passing through this vertex and the circumcircle of the triangle in two points which, with the other two vertices of the triangle, determines a parallelogram.

SOLUTION BY C. H. CHEPMELL, Hove, Sussex, England

Let ABC be a given triangle, with base $BC(=2\alpha)$, bisected in D; ABC, its circumcircle; FAE its Apollonian circle, cutting BC in E and F. Then AF and AE are, respectively, the external and internal bisectors of the angle A, cutting CB produced in F and E. Let AD cut the Apollonian circle in P and the circumcircle in H. Since F, B, E, C form an harmonic set of points, $\alpha^2=DE\cdot DF$. Also $DE\cdot DF=DP\cdot DA$ and $\alpha^2=HD\cdot DA$, from the elementary properties of circles. Hence $DP=HD$, and BC, PH bisect one another at D. Hence these last two lines are diagonals of $PCHB$, which is therefore a parallelogram.

NOTE: If the angle BAC is acute, P falls inside the triangle ABC; but if it is a right angle, P coincides with A; and, if it is obtuse, P falls outside ABC.

The cases, where ABC is isosceles or equilateral, have an interest of their own; but their adequate discussion might take us too far afield.

Also solved by RUTH A. BALLIEFF, MAX CORAL, J. W. CLAWSON, A. G. CLARK, H. A. DOBELL, T. C. ESTY and H. K. JUSTICE.

3235 [1927, 45]. Proposed by Paul Capron, U. S. Naval Academy.

If the curtate cycloid, $x=a\phi-b\sin\phi$, $y=a-b\cos\phi$, $(b>a)$, has a node at which $x=0$ when $\phi=\alpha$, and if h_1, h_2, and $-h_3$ are its maximum ordinate, the ordinate of a node, and its minimum ordinate, show that the area of a loop is $\alpha(h_1h_3-ah_2)$.

If the loops are just large enough to be tangent externally, show that the area of a loop is about $39.2a^2=8.52ab=1.85b^2$.

Solution by R. P. Agnew, Cornell University

If A is the area of a loop, then an expression for the area of the half-loop which lies to the left of the y-axis is

$$\frac{A}{2} = -\int_{\phi=0}^{\phi=\alpha} x\,dy = -\int_{\phi=0}^{\phi=\alpha} (ab\phi\sin\phi - b^2\sin^2\phi)\,d\phi,$$

where α is the positive value of the parameter ϕ which corresponds to the node on the y-axis. Then

$$A = 2ab(\alpha\cos\alpha - \sin\alpha) + b^2(\alpha - \sin\alpha\cos\alpha).$$

Since $b\sin\alpha=a\alpha$, this reduces to $A=\alpha(ab\cos\alpha-2a^2+b^2)$. Since $b\cos\alpha=a-h_2$, $a+b=h_1$, and $b-a=h_3$, it follows immediately that $A=\alpha(h_1h_3-ah_2)$.

Imposing the condition that adjacent loops be tangent externally, it is found that $a/b=\cos\psi$, where ψ is the positive first-quadrant solution of the equation $\tan\psi=\psi+\pi$. Then $\psi=1.35182$, and $a/b=.21724$. Solving the equation $\sin\alpha=.21724\alpha$ for α, we find $\alpha=2.5536$, and $\cos\alpha=-.8320$. Substituting the values of α, $\cos\alpha$, a/b, and b/a in the general expression for A, we find $A=39.2a^2=8.52ab=1.85b^2$. All the numerical results are approximations.

Also solved by E. M. Berry, A. G. Clark, J. B. Reynolds, and the Proposer.

3237 [1927, 98]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

A variable chord of an ellipse subtends a right angle at the center of the curve. Find the envelope of the chord.

Solution by R. E. Gaines, University of Richmond

Let κ be the length of the perpendicular from the center to the chord, and ρ_1, ρ_2, the lengths of the semi-diameters to the ends of the chord. From the right triangle, it follows that $\kappa^{-2}=\rho_1^{-2}+\rho_2^{-2}$; and, from the properties of the ellipse, the expression on the right is equal to $a^{-2}+b^{-2}$. Hence κ is a constant and the envelope is a circle with its center at the center of the ellipse and a radius $\kappa=ab/(a^2+b^2)^{1/2}$.

Also solved by R. P. Agnew, Theodore Bennett, H. C. Bradley, A. G. Clark, Michael Goldberg, J. E. Williams and R. M. Winger.

3238 [1927, 98]. Proposed by Malcolm Foster, Williams College.

Determine the necessary and sufficient condition that the loci of the centers of first and second curvature of a given curve have orthogonal tangents at corresponding points.

Solution by Theodore Bennett, University of Illinois

Let the coordinates of any point (x, y, z) on the original curve be expressed in terms of the arc length s. Then the loci of the centers of first and second curvature may be written

$$X_1 = x + \rho l, \qquad Y_1 = y + \rho m, \qquad Z_1 = z + \rho n,$$

$$X_2 = x + \tau\lambda, \qquad Y_2 = y + \tau\mu, \qquad Z_2 = z + \tau\nu,$$

respectively, using the notation given in Eisenhart's *Differential Geometry*, pp. 9–17. By differentiating these equations, and simplifying by means of the Frenet-Serret formulas, we find

$$X_1' = l\rho' - \lambda\rho/\tau, \qquad X_2' = \alpha + l + \lambda\tau',$$

and similar expressions for Y and Z. For the tangents to be orthogonal the necessary and sufficient condition is

$$X_1'X_2' + Y_1'Y_2' + Z_1'Z_2' \equiv \rho' - \rho\tau'/\tau = 0.$$

By integrating this last equation we find at once that $\rho = C\tau$, whence the radii of first and second curvature must have a constant ratio, *i.e.* the original curve must have been a cylindrical helix. See Eisenhart's Differential Geometry, pp. 20, 21.

Also solved by the PROPOSER.

3240 [1927, 98]. Proposed by Lü Ling, Tsing Hua University, Peking.

On a given base, construct a triangle such that its vertex may be on a given straight line, and the difference between its base angles may be equal to a given angle.

SOLUTION BY THEODORE BENNETT, University of Illinois

Let AB be the given base and let the angle at B exceed the one at A by θ. It is known from geometry that the third vertex P describes an equilateral hyperbola through A and B with center at O the mid-point of AB, with parallel tangents at A and B making an angle $\pi - \theta$ with AB, and with asymptotes making angles $(\pi-\theta)/2$ and $\pi - (\theta/2)$ with AB. A standard construction of projective geometry enables us to find the points in which this hyperbola is cut by the given line on which the third vertex must lie. The construction is modified only in minor details by the fact that some of the points are coincident at infinity.

NOTE BY OTTO DUNKEL

It is not necessary to consider the locus of P. The projective pencils AP and BP cut the given line l in two projective ranges of points, and we have to find on l the self-corresponding points of the two ranges. Construct on l the mates B_1, B_2, B_3 to three of its points A_1, A_2, A_3 chosen in any way. Draw a circle in the plane, and join any fixed point M of the circle with these six points. Let the six rays thus determined cut the circle in $\overline{A}_1$, $\overline{A}_2$, $\overline{A}_3$, $\overline{B}_1$, $\overline{B}_2$, $\overline{B}_3$. Draw the straight line m joining $(\overline{A}_1\overline{B}_2, \overline{A}_2\overline{B}_1)$ with $(\overline{A}_1\overline{B}_3, \overline{A}_3\overline{B}_1)$. If the line m cuts the circle in I, then the intersection of MI and l gives the desired third vertex of the triangle. There are two solutions of the problem if m cuts the circle in two distinct points; one solution if m is tangent to the circle; and no solution if it does not cut the circle.

3241 [1927, 98]. Proposed by F. A. Berger, Washington University.

With a center on the circumference of a given circle an arc of a second circle is drawn which bisects the area of the first circle. Find the ratio of the radii of the two circles.

SOLUTION BY J. E. WILLIAMS, Virginia Polytechnic Institute

Draw a circle with its center at O and a radius equal to r. Let A be one end of a diameter. With A as a center draw a circle with radius R, cutting the first circle at the points B and K, and let H be the mid-point of the arc of the circle A which lies within the circle O. Let the angle AOB be represented by ϕ and the angle OAB by θ.

One-half of the area between the circles is made up of the sector $HAB = \frac{1}{2}R^2\theta$ and the difference between the sector $AOB = \frac{1}{2}r^2\phi$ and the triangle $AOB = \frac{1}{2}Rr\sin\theta$. Since the area of the first circle is to be bisected, we have

$$R^2\theta + r^2\phi - Rr\sin\theta = \tfrac{1}{2}\pi r^2.$$

We also have $\phi = \pi - 2\theta$ and $R/r = 2\cos\theta$. With the aid of these two results the above equation reduces to

$$\sin 2\theta - 2\theta\cos 2\theta - (\pi/2) = 0.$$

It is readily seen from a figure that there is only one root between 0 and $\pi/2$ and that it must be less than $\pi/3$. By approximation processes it is found that

$$\theta = 54^\circ\,35'\,38'', \qquad R/r = 1.15874.$$

Also solved by THURMAN ANDREW, THEODORE BENNETT, MICHAEL GOLDBERG, A. PELLETIER, and S. D. TURNER.

3242 [1927, 98]. Proposed by R. S. Underwood, Auburn, Alabama.

A man finds that a pile of cocoanuts is exactly divisible by n after giving an extra nut to a monkey. He takes away $1/n$th of the remaining nuts and leaves the rest. A second man repeats the process with the rest giving one nut to the monkey and taking away exactly $1/n$th of the rest. This is continued and the nth man leaves at the end a pile of nuts which is exactly divisible by n. How many nuts were there at the beginning and how many at the end?

SOLUTION BY ROBERT E. MORITZ, University of Washington

Let x equal the number of nuts at the beginning, y, the number of nuts at the end. Then by the conditions of the problem,

(1) $$x-1=kn, \quad y=hn, \qquad \text{where } k, h, \text{ and } n, \text{ are unknown integers.}$$

Denote $(n-1)/n$ by p. Then the number left by the first man is knp, while the number left by the rth man is

(2) $$knp^r - (p^{r-1} + p^{r-2} + \cdots + p^2 + p) = \frac{(k+1)(n-1)^r - (n-1)n^{r-1}}{n^{r-1}}.$$

Setting this last expression equal to hn for $r=n$, we have

(3) $$(k+1)/[(h+1)n - 1] = n^{n-1}/(n-1)^n,$$

that is, $k+1$ and $(h+1)n-1$ must be equi-multiples of n^{n-1} and $(n-1)^n$ respectively, sav

(4) $$k+1 = mn^{n-1}, \quad (h+1)n - 1 = m(n-1)^n.$$

The second of the equations (4) may be written in the form

(5) $$h = (n-1)n^{-1}[m(n-1)^{n-1} - 1]$$

from which it appears that $m(n-1)^{n-1}-1$ must be a multiple of n. It is evident also that if one value of m say m', has been found such that $m'(n-1)^{n-1}-1$ is a multiple of n, the most general value of m which will make h integral is $m=m'+nc$, where c is any integer. If m' is the smallest positive integral value of m, all possible integral values of h are found by giving to c the successive values $1, 2, 3, \cdots$. The first of equations (4) determines the corresponding values of k, and (1) the values of x and y. The general solution corresponding to any given value of n is thus found to be

(6) $$x = (m' + nc)n^n - (n-1), \qquad y = (m' + nc)(n-1)^n - (n-1),$$

where m' is the smallest positive integer that will make $m'(n-1)^{n-1}-1$ a multiple of n.

Now, on expanding the expression $(n-1)^{n-1}$ in powers of n, we see that the least positive value of m which will satisfy the congruence

$$m(n-1)^{n-1} - 1 \equiv 0, \mod n,$$

is $m'=1$ when n is odd, and $m'=n-1$ when n is even. We have therefore the general solution of our problem in the form

(7)
$$x = (1+nc)n^n - (n-1), \qquad y = (1+nc)(n-1)^n - (n-1), \quad n \text{ odd},$$
$$x = (n-1+nc)n^n - (n-1), \qquad y = (n-1+nc)(n-1)^n - (n-1), \quad n \text{ even},$$

and these two forms may be combined into the single form

(8) $$x = Kn^n - (n-1), \quad y = K(n-1)^n - (n-1), \quad 2K = n(1+2c) + (-1)^{n+1}(2-n),$$

in which n and c are any two positive integers.

The following table gives the numerical values of both the principal solutions ($k=k'$), and the general solutions of x and y for all values of n from $n=2$ to $n=10$.

n	m'	x_0	y_0	x_c	y_c
2	1	3	0	$3+8c$	$0+2c$
3	1	25	6	$25+81c$	$6+24c$
4	3	765	240	x_0+4^5c	$y_0+4.3^4c$
5	1	3,121	1,020	x_0+5^6c	$y_0+5.4^5c$
6	5	233,275	78,120	x_0+6^7c	$y_0+6.5^6c$
7	1	823,537	279,930	x_0+7^8c	$y_0+7.6^7c$
8	7	117,440,505	40,353,600	x_0+8^9c	$y_0+8.7^8c$
9	1	387,420,481	134,217,720	$x_0+9^{10}c$	$y_0+9.8^9c$
10	9	89,999,999,991	31,381,059,600	$x_0+10^{11}c$	$y_0+10.9^{10}c$

Also solved by R. P. AGNEW, H. C. BRADLEY, E. M. BERRY, P. S. DWYER, L. H. FELTMAN, R. E. GAINES, J. S. GEORGES, MICHAEL GOLDBERG, H. O. HANSON, P. A. KNEDLER, JOSHUA MATZ, RICHARD MORRIS, J. R. MUSSELMAN, A. PELLETIER, B. C. PATTERSON, J. E. ROWE, G. E. RAYNOR, L. S. SHIVELY, F. L. WILMER, and the PROPOSER.

3247[1927, 156]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

When can two conics be projected into two orthogonal circles by a real projection?

SOLUTION BY THE PROPOSER.

The operation being projective, and therefore reversible, we may try to answer the proposed question by considering the contrary problem: *If two orthogonal circles are transformed by a real projection, in what relation will the resulting conics be to each other?* Or, in other words, do two orthogonal circles have a characteristic property which is projective?

It is known that the radical axis of two orthogonal circles has for its pole with respect to either of the circles the center of the other cricle. This property holds when both circles are real, or when one circle is real and the other imaginary. Let us now establish the converse of this proposition.

If the radical axis r of two circles (A), (B) is the polar of the center A of one of the circles with respect to the other circle (B), then: (a) the radical axis r is also the polar of the other center B with respect to the first circle (A); (b) the two circles are orthogonal.

Let a, b be the radii of the two circles, and R the point of intersection of the radical axis with the line of centers AB. The point R has the same power with respect to both circles; hence we have, both in magnitude and in sign, (1) $RA^2-a^2=RB^2-b^2$. On the other hand R, A are, by assumption, inverse points with respect to the circle (B); hence we have, again both in magnitude and in sign, (2) $BA\,.\,BR=b^2$. The relation (1) may be written, due regard being paid to the signs of the segments involved,

$$a^2-b^2=RA^2-RB^2=(RA+RB)(RA+BR)=(RA+RB)BA.$$

By adding this result to (2) we get $(RA+RB+BR)BA=a^2$, or (3) $AR\,.\,AB=a^2$. Thus the points B, R are inverse points with respect to the circle (A), which proves part (a) of the proposition.

Adding (2) and (3) we have

$$a^2+b^2=AR\,.\,AB+BA\,.\,BR=AR\,.\,AB+AB\,.\,RB=AB(AR+RB)=AB^2,$$

which proves part (b) of the proposition.

The radical axis r is a common chord of the two circles and joins two real common points of the two curves, if both circles are real, or a pair of conjugate imaginary common points of the two curves, if one of the circles is imaginary but has a real equation. The two circles have also in common the circular points at infinity of the plane, and the line at infinity is a real common chord of the two circles, and

furthermore the centers of the two circles are the poles of this line with respect to these circles. Thus the property of two orthogonal circles quoted above may be formulated in projective terms. We are therefore able to answer the proposed question as follows: Two conics (P), (Q), one of which, at least, is real, may be transformed into two orthogonal circles by a real projection, if they have two real common chords, s, t such that the pole of one of these chords, say s, with respect to one of the conics, say (P), coincides with the pole of the other chord t with respect to the other conic (Q).

If both conics are real one of the chords must join a pair of real common points and the other a pair of conjugate imaginary common points of the two conics.

REMARK: Two conics related in this manner will have all the descriptive properties of two orthogonal circles. Many of these properties become particularly interesting in their new application. For lack of space we shall consider only one example. It is known that the lines on which two conics determine two pairs of harmonic points envelop, in general, a conic. In the case of two conics which may be transformed into two orthogonal circles this envelope consists of two points, namely, the two points in which coincide the four poles of their common chords s, t.[1]

EXTENSION TO SPACE: The proposition proved above about two orthogonal circles applies without modification to two orthogonal spheres. The point R will be the trace on the line of centers of the radical plane of the two spheres. We may therefore extend our result to two orthogonal spheres. Thus we have the proposition: *Two quadric surfaces (P), (Q), at least one of which is real, may be transformed into two orthogonal spheres by a real projection if they have two real intersections in planes σ, τ such that the pole of one of these planes, say σ, with respect to one of the quadrics, say (P), coincides with the pole of the other plane τ with respect to the other surface (Q).*

If both surfaces are real, one of the common planes must cut the surfaces along a real conic and the other along an imaginary conic.

Also solved by THEODORE BENNETT and PAUL WERNICKE.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Nashville meeting was a great success. The attendance was large and representative. The South did itself proud both in numbers and in active support. But the North and the East and the West were all in evidence also, both on the programs and at the joint dinner. Speaking of dinner, were there ever such meals served at such moderate prices as Ward-Belmont provided! We all vote for another meeting in the South.

The first volume of the Ahmes Mathematical Papyrus is now printed and bound, but the second volume is still delayed on account of tedious work connected with the perfecting of the colored plates. We are assured, however, that these difficulties are now practically surmounted, and that the completion of the second volume may confidently be hoped for sometime in March. The two volumes will be shipped together as soon as they are ready.

The following schedule of lectures has been arranged for Professor CONSTANTIN CARATHÉODORY of the University of Munich who will be the first

[1] Nathan Altshiller-Court, *College Geometry* (Johnson Pub. Co., Richmond, Va., 1925), p. 148 ex. 15.

Visiting Lecturer of the American Mathematical Society. The University of Pennsylvania, January 10–13; Ohio State University, January 16–17; The University of Iowa, January 19–20; The University of Chicago, January 23–24; The University of Michigan, January 25–27; Adelbert College, January 30–31; Cornell University, February 1–2. After these lectures, Professor Carathéodory will be in residence at Harvard University until about June 1.

The Board of Managers of the Franklin Institute has voted to award the Elliott Cresson Gold Medal to Professor VLADIMIR KARAPETOFF of Cornell University in consideration of the inventive ability, skill in design, and detailed theoretical knowledge of kinematics and electrical engineering displayed in the development of computing devices.

At the Bartol Research Foundation, Professor EINAR HILLE of Princeton University lectured on December 9 and 15, 1927 on "Boundary problems in differential equations with special reference to their application to Schroedinger's wave equation"; and Professor ARNOLD DRESDEN of Swarthmore College gave a lecture on January 10, 1928, on "Special devices used in the solution of problems by the matrix mechanics."

At the Nashville Meeting of the American Association for the Advancement of Science, Professor R. C. ARCHIBALD of Brown University was elected Vice-president and Chairman of Section A, and Professor C. N. MOORE of the University of Cincinnati was elected Secretary. At the same meeting Professor D. R. CURTISS of Northwestern University was elected a member of the Executive Committee.

Dr. HAROLD HOTELLING, research associate in the Food Research Institute until the fall of 1927, has been appointed associate professor of mathematics at Stanford University.

Dr. G. T. WHYBURN has been appointed assistant professor of mathematics at the University of Texas.

Professor J. L. RILEY, Ouachita College, a charter member of the Mathematical Association, died on July 12, 1927, from the effects of an operation.

Dr. G. A. OSBORNE, emeritus professor of mathematics at the Massachusetts Institute of Technology, died on November 20, 1927, at the age of eighty-eight years. He held membership in the Mathematical Association from December, 1916, to his death.

Professor C. D. GARLOUGH of Wheaton College, a member of the Mathematical Association since 1920, died on November 25, 1927.

THE NOVEMBER MEETING OF THE MISSOURI SECTION

The eleventh annual meeting of the Missouri Section of the Mathematical Association of America was held at Washington University, St. Louis, Missouri, on Friday and Saturday, November 25–26, 1927, in connection with the meeting of the Southwestern Section of the American Mathematical Society. A reception was held at the University Club on Friday evening under the auspices of the Washington University Chapter of the Pi Mu Epsilon Mathematical Fraternity. At this reception several musical numbers were given, and Professor E. B. Stouffer related some of his experiences in Italy, particularly at the University of Bologna, while engaged in research work as the holder of a Guggenheim Fellowship. A session held at nine o'clock Saturday morning was presided over by Miss Kathryn Wyant, chairman of the section, and by Professor W. H. Roever, vice-chairman, who occupied the chair while Miss Wyant presented a paper. This session was followed by a session of the Southwestern Section of the Society at ten-thirty. The Saturday afternoon session, which was a joint session with the Southwestern Section, was presided over by Professor Roever. Those attending the meetings were the guests of Washington University at luncheon on Saturday.

The attendance was about thirty, including the following twenty-two members of the Association: Florence Black, W. C. Brenke, S. C. Davisson, Otto Dunkel, F. J. Gerst, Cornelius Gouwens, L. D. Haertter, Byron Ingold, Louis Ingold, C. G. Jaeger, Harry Levy, Jesse Osborn, W. O. Pennell, P. R. Rider, G. Edna Robinson, W. H. Roever, Eugene Stephens, E. B. Stouffer, G. B. Sweazey, J. S. Turner, E. Kathryn Wyant, Jessica M. Young.

The following officers were elected for 1928: Chairman, W. A. LUBY, Kansas City Junior College; Vice-chairman, G. B. SWEAZEY, Westminster College; Secretary-Treasurer, P. R. RIDER, Washington University. The 1928 meeting will be held in Kansas City at the time of the meeting of the Missouri State Teachers Association in November.

The following two papers were presented at the meeting of the Missouri Section:

1. "The foundation of the theory of ideals" by Miss KATHRYN WYANT, University of Missouri.

2. "The trigonometry of hyperspace" by Miss NOLA LEE ANDERSON, University of Missouri. (By invitation.)

At the joint session Professor E. B. STOUFFER of the University of Kansas presented a paper on "Some canonical forms and associated canonical expansions in projective differential geometry" (by invitation of the program committee).